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ON THE EXISTENCE OF JENKINS–STREBEL DIFFERENTIALS

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Abstract

Let S be a compact Riemann surface of genus g > 1. By using Teichmüller theory, a new approach is obtained to the existence of a Jenkins–Strebel differential with prescribed type and heights on S. Moreover, the geometric structures of horizontal trajectories of certain classes of quadratic differentials are discussed.

Introduction

Let S_0 be a smooth orientable closed surface of genus $g > 1$. The set of conformal structure classes that are not conformally equivalent on S_0 is called the *Riemann moduli space*, denoted by $R(S_0)$. An important problem is to give $R(S_0)$ a parametrization. A significant contribution towards this problem was made by O. Teichmüller. In the famous papers [**30**, **31**], he first considered the set of isotopy classes of conformal structures on S_0. Now this set is known as the *Teichmüller space* $T(S_0)$. By using extremal quasi-conformal mappings, Teichmüller proved that the space $T(S_0)$ carries a natural metric and is homeomorphic to the $(6g-6)$-dimensional Euclidean space $\mathbb{R}^{6g-6}$ in the metric topology. Using the fact that $T(S_0)$ is the ramified covering space of $R(S_0)$, Teichmüller obtained a parametrization of $R(S_0)$.

One of the essential tools in Teichmüller's proof is quadratic differentials theory. This made Teichmüller the first mathematician to study the geometric properties of holomorphic quadratic differentials; see [**30**, **31**]. Later, A. Marden and K. Strebel [**20**, **21**, **25**, **28**], F. P. Gardiner [**10**, **11**], and other mathematicians studied the geometric theory of quadratic differentials extensively. In [**17**, **18**], S. Kerckhoff studied the relationship between Teichmüller space theory and the local Euclidean geometry induced by the quadratic differentials. He also obtained some results on the trajectory structures of quadratic differentials by using the length-area method and the Thurston topology on measured foliations.

Some existence theorems for quadratic differentials with closed trajectories on a compact Riemann surface were first obtained by J. A. Jenkins [**14**] and K. Strebel [**25**, **26**], working from different viewpoints. J. Hubbard and H. Masur [**12**] and H. Renelt [**24**] simultaneously solved the 'height problem' via different methods. Their work shows that one can prescribe an admissible curves system Γ and the numbers $h_i > 0$. Then there exists a unique Jenkins–Strebel differential of type Γ on S, and the heights of its cylinders are the given numbers h_i (see Section 1).

The theory of quadratic differentials has long played a central role in the study of Teichmüller theory. However, in this paper we show that Teichmüller theory can also be useful in the study of quadratic differentials. Here we give a new

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proof of the 'height problem' from a Teichmüller theory viewpoint. Our work has the advantage that more geometric information on quadratic differentials can be obtained. Following [**12**, **17**], we are mainly concerned with the interplay between a conformal structure and the 'flat' metric structure induced from the Jenkins–Strebel differential on S. Therefore the surface S can be viewed as a united body of some Euclidean cylinders. In this way, it is convenient to deal with the trajectory structures of certain quadratic differentials.

The paper is composed of three sections. In Section 1 we define some terminology and recall some background. Section 2 is devoted to the proof of the main result. Other results are left to the last section.

1. *Background*

Let S_0 be a smooth orientable closed surface of genus g $(g>1)$, and let S be a compact Riemann surface of the same genus. The surface S_0 can be given the conformal structure S_σ by pulling back the conformal structure of S through the diffeomorphism $\sigma : S_0 \longrightarrow S$. In the following context, the Riemann surfaces S and the conformal structures S_σ on S_0 are terms used interchangeably.

Let the set $\tilde{T}(S_0)$ consist of all conformal structures on S_0. We define the equivalence relation '$\sim$' in $\tilde{T}(S_0)$ as $S_{\sigma_1} \sim S_{\sigma_2}$ if and only if there exists a holomorphic homeomorphism $h : S_{\sigma_1} \longrightarrow S_{\sigma_2}$ homotopic to the identity.

Definition 1.1. The *Teichmüller space* $T(S_0)$ is defined to be $T(S_0) = \tilde{T}(S_0)/\sim$. We denote by $[S_\sigma] \in T(S_0)$ the class of conformal structures equivalent to S_σ.

If S_1 and S_2 are two compact Riemann surfaces of genus g and $f : S_1 \longrightarrow S_2$ is a quasi-conformal homeomorphism, we denote by

$$K_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|}$$

the dilatation of f at z_0, where $\mu_f(z) = (\partial_{\bar z} f d\bar z)/(\partial_z f dz)$ is the Beltrami differential of f. Let $K_f = \operatorname{ess\,sup}_{z_0 \in S} K_f(z_0)$ be the maximal dilatation of f. In the Teichmüller space $T(S_0)$, the Teichmüller metric $d_{\mathrm{T}}(\cdot,\cdot)$ is defined to be

$$d_{\mathrm{T}}\big([S_{\sigma_1}], [S_{\sigma_2}]\big) = \sup_h \log K_h,$$

where the supremum is taken over all quasi-conformal homeomorphisms $h \simeq \mathrm{id} : S_{\sigma_1} \longrightarrow S_{\sigma_2}$. The Teichmüller space $T(S_0)$ is a complete metric space in the Teichmüller metric $d_{\mathrm{T}}(\cdot,\cdot)$; see [**13**, **19**].

Each non-degenerate ring domain R embedded in the Riemann surface S inherits a conformal structure from S. Therefore R is conformally equivalent to exactly one normal flat cylinder

$$\tilde{R} = \{z \mid 1 < |z| < e^r\}$$

in the z-plane, where $r > 0$.

Definition 1.2. With the above notation, the *modulus* of the ring domain $R \subset S$ is defined to be $r/2\pi$.

Let $Q(S)$ be the space of holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on S. It is a Banach space in the L^1-norm $\|\varphi\| = \iint_S |\varphi| dx\, dy$. The Riemann–Roch theorem shows that $Q(S)$ has complex dimension $3g-3$.

Each non-zero quadratic differential φ induces a singular metric $ds = \sqrt{|\varphi(z)|}\, |dz|$. The φ-length of any piecewise smooth curve $\gamma \subset S$ is defined to be

$$l_\varphi(\gamma) = \int_\gamma \sqrt{|\varphi(z)|}\, |dz|.$$

A horizontal trajectory of φ is a maximal curve along which $\varphi > 0$, and a vertical trajectory of φ is a maximal curve along which $\varphi < 0$. A trajectory is *critical* if it meets a singularity (zero) of φ when continued in either direction; otherwise it is regular.

For any simple curve $\gamma \subset S$, it turns out to be important to consider the φ-height of γ, as well as the φ-length.

Definition 1.3. Let $\varphi \in Q(S)$ be a non-zero holomorphic quadratic differential. For any piecewise smooth curve $\gamma \subset S$, the infimum

$$h_\varphi(\gamma) = \inf_{\tilde\gamma \sim \gamma} \int_{\tilde\gamma} |\Im \sqrt{\varphi}|,$$

where $\tilde\gamma$ varies over all piecewise smooth curves in the homotopy class of γ, is called the *height* of γ with respect to φ.

As in [**29**], we give the following definition.

Definition 1.4. A non-zero quadratic differential $\varphi \in Q(S)$ is called a *Jenkins–Strebel differential* if its non-closed trajectories cover a set of measure zero.

For any Jenkins–Strebel differential φ, Strebel [**29**] gave the following criterion: φ is a Jenkins–Strebel differential if and only if its critical graph $\mathcal{C}$ (the set of critical trajectories and their critical endpoints) is compact. Thus the set $S \setminus \mathcal{C}$ consists of a collection of cylinders $\{R_k\}$ swept out by closed horizontal trajectories. We call $\{R_k\}$ the *characteristic cylinders* of the Jenkins–Strebel differential φ.

Basic reference sources for the theory of quadratic differentials are [**11**, **29**].

A system of finitely many smooth closed curves $\{\gamma_1, \gamma_2, \dots, \gamma_p\} \subset S_0$ is called *admissible* if none of the curves γ_i is homotopically trivial (homotopic to zero), and if the two curves γ_i and γ_j neither intersect nor are freely homotopic for $i \neq j$. The maximal number of curves in an admissible system on S_0 is $3g-3$; see [**11**, Section 10.1]. A ring domain $R_0 \subset S_0$ is said to be *of homotopy type* γ if there is a closed curve $\gamma_0 \subset R_0$ that separates its two boundary components and is freely homotopic to γ.

Let $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ be an admissible system on S_0. A family of non-overlapping ring domains $\{R_1, R_2, \dots, R_p\} \subset S_0$ is said to be *of homotopy type* $\{\gamma_k\}$ if each R_k is of homotopy type γ_k for exactly one k.

Definition 1.5. Let $\{\gamma_k\}$ be an admissible system on S_0. A Jenkins–Strebel differential φ is said to be of *type* $\{\gamma_k\}$ if its characteristic cylinders $\{R_k\}$ have homotopy type $\{\gamma_k\}$.

Let R be a characteristic cylinder of a Jenkins–Strebel differential φ. For any simple vertical arc γ connecting the two boundary components of R, it is easy to check that the value $h = \int_\gamma |\Im\sqrt{\varphi}|$ is independent of γ. We call h the φ-height of R, denoted by $h_\varphi(R)$.

The following used to be a famous problem. It was simultaneously solved by Hubbard and Masur [**12**] and by Renelt [**24**].

THEOREM 1.6 [**12**, **24**]. *Let $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ be an admissible curves system on a compact Riemann surface S. Then, for arbitrary $\{h_k > 0, k = 1, 2, \dots, p\}$, there exists a Jenkins–Strebel differential φ with type $\{\gamma_k\}$ and φ-heights $\{h_k\}$. Moreover, φ is uniquely determined.*

In Sections 2 and 3, we will present a new proof of Theorem 1.6 by using Teichmüller theory. This shows that Teichmüller theory can also be useful in the study of quadratic differentials.

2. *Proofs of the main results*

Let $\mathcal{P}$ be a smooth bordered surface of signature $(0, 3)$; that is, it is obtained from a sphere by cutting away the interiors of three disjointed closed disks. We label its border components by γ_1, γ_2 and γ_3, respectively.

LEMMA 2.1. *Let (h_1, h_2, h_3) be a fixed triple of positive numbers. For any positive triple (l_1, l_2, l_3), the surface $\mathcal{P}$ can carry a conformal structure such that the resulting bordered Riemann surface P has the following properties.*

(i) *The Riemann surface P admits a Jenkins–Strebel differential φ of type $\{\gamma_1, \gamma_2, \gamma_3\}$ and the boundary components γ_1, γ_2, and γ_3 are all closed horizontal trajectories of φ.*

(ii) *In the φ-metric, the characteristic cylinders $R_i \subset P$ have circumferences l_i and heights h_i, where $i = 1, 2, 3$.*

Proof. Performing a 'cut' along the negative axis, we can always select the single-valued branch of the analytic function $z = z(\zeta) = \zeta^q$ (where q is a rational number) so that $z(1) = 1$.

Given the triple (h_1, h_2, h_3), for any positive triple (l_1, l_2, l_3), let

$$D_i = \{\xi + i\eta \,|\, 0 \leqslant \xi \leqslant l_i/2, 0 \leqslant \eta \leqslant h_i/2\}, \qquad i = 1, 2, 3,$$

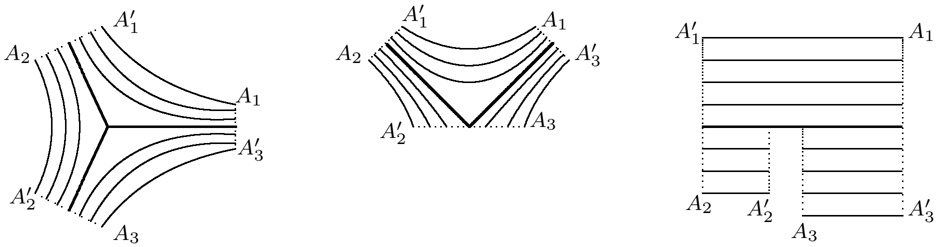
be the rectangles in the $(\zeta = \xi + i\eta)$-plane.

For simplicity, we assume that $l_1 = \max\{l_1, l_2, l_3\}$. Then the triple (l_1, l_2, l_3) satisfies one of the following.

(1) $l_1 < l_2 + l_3$. Let $A_1A_1'A_2A_2'A_3A_3' = \bigcup \tilde{D}_i$ be the 'hexagon' in the z-plane, where

$$\tilde{D}_1 = \left(\zeta - \frac{l_1 + l_2 - l_3}{4}\right)^{2/3} (D_1),$$

$$\tilde{D}_2 = e^{(2/3)\pi i} \left(\zeta - \frac{l_2 + l_3 - l_1}{4}\right)^{2/3} (D_2),$$



(a) (b) (c)

FIGURE 1.

and

$$\tilde{D}_3 = e^{(4/3)\pi i}\left(\zeta - \frac{l_3 + l_1 - l_2}{4}\right)^{2/3} (D_3)$$

as in Figure 1(a). We define $\varphi = (3/2)^2 z dz^2$, the quadratic differential in the 'hexagon' $A_1A_1'A_2A_2'A_3A_3'$.

(2) $l_1 = l_2 + l_3$. Set

$$\tilde{D}_1 = e^{(1/4)\pi i}\left(\zeta - \frac{l_2}{2}\right)^{1/2} (D_1),$$

$$\tilde{D}_2 = e^{(3/4)\pi i}\zeta^{1/2}(D_2),$$

and

$$\tilde{D}_3 = e^{(1/4)\pi i}\zeta^{1/2}(\overline{D}_3),$$

where $\overline{D}_3$ denotes the conjugation image of D_3.

As in Figure 1(b), we denote by $A_1A_1'A_2A_2'A_3A_3' = \bigcup \tilde{D}_i$ the 'hexagon' in the z-plane, and we define $\varphi = -4z^2dz^2$, the quadratic differential in the 'hexagon' $A_1A_1'A_2A_2'A_3A_3'$.

(3) $l_1 > l_2 + l_3$. Let $A_1A_1'A_2A_2'A_3A_3' = \bigcup \tilde{D}_i$ be the 'hexagon' in the z-plane, where $\tilde{D}_1, \tilde{D}_2$, and $\tilde{D}_3$ are three rectangles in the z-plane, as in Figure 1(c). In coordinate form

$$A_1 = \left(\frac{l_1}{2}, \frac{h_1}{2}\right), \quad A_1' = \left(0, \frac{h_1}{2}\right), \quad A_2 = \left(0, -\frac{h_2}{2}\right),$$

$$A_2' = \left(\frac{l_2}{2}, -\frac{h_2}{2}\right), \quad A_3 = \left(\frac{l_1 - l_3}{2}, \frac{h_3}{2}\right), \quad A_3' = \left(\frac{l_1}{2}, -\frac{h_3}{2}\right).$$

We define $\varphi = dz^2$, the quadratic differential in the 'hexagon' $A_1A_1'A_2A_2'A_3A_3'$.

The bordered Riemann surface $\tilde{P} = A_1A_1'A_2A_2'A_3A_3'$ has a mirror image $\tilde{P}^*$. The image $\tilde{P}^*$ can be glued along the border components $\{A_1'A_2, A_2'A_3, A_3'A_1\}$ to the original surface $\tilde{P}$ to form a new surface P. The surface P is called the *double* of $\tilde{P}$ along $\{A_1'A_2, A_2'A_3, A_3'A_1\}$. Supposing that z is any local coordinate parameter of $\tilde{P}$, then $z^* = \bar{z}$ is a local coordinate parameter of $\tilde{P}^*$. In terms of the local parameter z^*, the quadratic differential $\varphi = \varphi(z)dz^2$ on $\tilde{P}$ induces the quadratic differential

$$\varphi^* = \varphi^*(z^*)\, dz^{*2} = \overline{\varphi(\overline{z^*})}\, d\bar{z}^2$$

on $\tilde{P}^*$. Since the set $\{A_1'A_2, A_2'A_3, A_3'A_1\}$ consists of only horizontal lines or vertical lines of φ, two quadratic differentials φ and φ^* coincide on $\{A_1'A_2, A_2'A_3, A_3'A_1\}$. By analytic continuation, φ and φ^* can be joined into a new quadratic differential on P, denoted by the same name φ.

For $i = 1, 2, 3$, the arc A_iA_i' and its mirror image together form the border component γ_i; these are closed horizontal trajectories of φ. From the construction, the new surface P has the desired properties.

We call A_i the *marked point* of γ_i on P, $i = 1, 2, 3$ respectively. □

From now on, let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{3g-3}\}$ be a maximal finite admissible curves system on S_0. Cutting along Γ, we divide S_0 into $2g-2$ topological surfaces of signature (0,3) and label them by $\{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{2g-2}\}$.

In [**5**], P. Buser introduced the notion of cubic graphs.

A graph $\mathcal{G}$ is called 3-*regular* if each of its vertices has three emanating edges. For our purpose, it is useful to view each edge of the graph as a union of two half-edges emanating from one of the two connected vertices.

DEFINITION 2.2. A *cubic graph* $\mathcal{G}$ is a finite 3-regular connected graph. It will be taken into consideration when welding pairs of pants into a compact Riemann surface.

For each $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{2g-2}\}$, we label its border components by $\{\gamma_{i1}, \gamma_{i2}, \gamma_{i3}\}$. Given the maximal admissible system Γ, the cubic graph $\mathcal{G}_\Gamma$ associating to Γ is a graph whose $2g-2$ vertices $\{y_i\}$ are in one-to-one correspondence with the $2g-2$ surfaces $\{\mathcal{P}_i\}$, the $3g-3$ edges $\{c_k\}$ correspond to the $3g-3$ closed curves $\{\gamma_k\}$, and three emanating half-edges $\{c_{i\mu}\}$ of y_i appearing in the graph correspond to the three border components $\{\gamma_{i\mu}\}$ of γ_i.

In this way, if $c_{i\mu}$ and c_{jv} are two half-edges of the edge c_k, we write $c_k = (c_{i\mu}, c_{jv})$. Therefore the cubic graph $\mathcal{G}_\Gamma$ can be fully described by a list $c_k = (c_{i\mu}, c_{jv})$, where $k = 1, 2, \dots, 3g-3$. When constructing the Riemann surfaces, it is practical to view the list itself as a cubic graph.

Now suppose that the positive array $H = (h_1, h_2, \dots, h_{3g-3})$ serves as heights with respect to the admissible system Γ.

Let $v = (L_v, A_v) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, where

$$L_v = (l_1, l_2, \dots, l_{3g-3}) \in \mathbb{R}_+^{3g-3}, \qquad A_v = (\theta_1, \theta_2, \dots, \theta_{3g-3}) \in \mathbb{R}^{3g-3}.$$

For any $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{2g-2}\}$, by Lemma 2.1, there exists a conformal structure on $\mathcal{P}_i$ so that the resulting Riemann surface P_i admits a Jenkins–Strebel differential φ_i, whose characteristic cylinders have circumferences $\{l_{i\mu}\}$ and heights $\{h_{i\mu}/2\}$ in the φ_i-metric.

For $\mu = 1, 2, 3$, as in the proof of Lemma 2.1, we label by $\zeta_{i\mu}$ the marked point of $\gamma_{i\mu}$ on P_i. In the singular φ_i-metric, the horizontal curve $\gamma_{i\mu}$ can be parametrized with constant speed by

$$t \to \gamma_{i\mu}(t), \quad t \in [0, 1] \quad \text{and} \quad \gamma_{i\mu}(0) = \gamma_{i\mu}(1) = \zeta_{i\mu},$$

where the increasing direction of the parameter t is consistent with the orientation of $\gamma_{i\mu}$. This parametrization can be also interpreted as a parametrization of $\gamma_{i\mu}$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ instead of $[0, 1]$.

The conformal structure $h_H(v)$ on S_0 is defined as follows. For each list $c_k = (c_{i\mu}, c_{jv})$ in the cubic graph $\mathcal{G}_\Gamma$, the pairs of pants P_i and P_j must be taken into account. Their border components satisfy

$$l_k = l_{\varphi_i}(\gamma_{i\mu}) = l_{\varphi_j}(\gamma_{jv}), \quad \text{for } k = 1, 2, \dots, 3g-3.$$

Then we can weld pairs of pants P_i and P_j together along the border curves $\gamma_{i\mu}$ and γ_{jv} by identifying points $\gamma_{i\mu}(t)$ and $\gamma_{jv}(\theta_k - t)$ for all $t \in \mathbb{S}^1$. The resulting Riemann surface is denoted by $h_H(v)$. Since the curve $\gamma_{i\mu}$ is a horizontal closed trajectory of φ_i and γ_{jv} is a horizontal closed trajectory of φ_j, this welding is possible. The $2g-2$ quadratic differentials $\{\varphi_i\}$ are joined into a Jenkins–Strebel differential φ_H on the Riemann surface $h_H(v)$, with type Γ, and the characteristic cylinders $\{R_k\}$ have φ_H-heights $\{h_k\}$.

These gluing operations allow us to define a mapping:

$$h_H : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow T(S_0),$$
$$h_H(v) = [h_H(v)].$$

The construction shows that the mapping $h_H : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow T(S_0)$ is well defined.

The main result of this paper is the following theorem.

THEOREM 2.3. *Given a maximal admissible system* Γ *on* S_0*, for any positive array* $H = (h_1, h_2, \dots, h_{3g-3})$*, the mapping* $h_H : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow T(S_0)$ *is a homeomorphism.*

The proof of Theorem 2.3 will be deferred to Section 3. We deduce Theorem 1.6 from Theorem 2.3 first.

Proof of Theorem 1.6. Let $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ be an admissible system on S and $(h_1, h_2, \dots, h_p)$ be the heights corresponding to $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$. Then $p \leqslant 3g-3$.

If $p = 3g-3$, we set $H = (h_1, h_2, \dots, h_{3g-3})$. Theorem 2.3 shows that the mapping

$$h_H^{-1} : T(S_0) \longrightarrow \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

is a homeomorphism. By using the data $h_H^{-1}([S])$, we obtain a Jenkins–Strebel differential $\varphi_H \in Q(S)$ of type $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$, and its characteristic cylinder R_k has height h_k, where $1 \leqslant k \leqslant p$.

If $p < 3g-3$, then $3g-3-p$ additional simple closed curves $\{\gamma_{p+1}, \dots, \gamma_{3g-3}\}$ can be added to $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ so that $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_p, \gamma_{p+1}, \dots, \gamma_{3g-3}\}$ is a maximal admissible system on S_0; see [**11**, Section 10.1].

For any positive vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{3g-3-p})$, there exists a Jenkins–Strebel differential $\varphi_\varepsilon \in Q(S)$ of type Γ and heights $(h_1, \dots, h_p, \varepsilon_1, \dots, \varepsilon_{3g-3-p})$. Now, assuming that $\varepsilon \to 0+$, we claim that the norms $\|\varphi_\varepsilon\|$ are uniformly bounded. Hence as $\varepsilon \to 0+$, the quadratic differentials φ_ε converge uniformly to a quadratic differential φ, with the prescribed type $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$, and its characteristic cylinders have heights $(h_1, h_2, \dots, h_p)$.

In order to show that the norms $\|\varphi_\varepsilon\|$ are uniformly bounded, we review the following lemma in [**29**] about the minimal sum of the weighted sum of the reciprocals of moduli. This lemma plays a crucial role throughout this paper. For the sake of completeness, we present a simple proof here. $\square$

Lemma 2.4. *Let $\varphi \neq 0$ be a Jenkins–Strebel differential of type $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$, and suppose that its characteristic cylinder $\{R_k\}$ has heights $\{h_k\}$ on S. Let $\tilde{R}_k$ be a system of non-overlapping ring domains on S with the homotopy type $\{R_k\}$; then their moduli $\tilde{M}_k$ satisfy*

$$\sum \frac{h_k^2}{\tilde{M}_k} \geqslant \sum \frac{h_k^2}{M_k},$$

with equality if and only if $\tilde{R}_k = R_k$ for each k.

Proof. If a degenerate ring domain $\tilde{R}_k$ occurs, then there is nothing to do. Therefore we assume that $\tilde{M}_k > 0$ for all k.

We map $\tilde{R}_k$, after cutting it radially, onto a horizontal rectangle $\tilde{S}_k$ of the z-plane. $\tilde{S}_k$ is normalized such that it has height h_k and length $\tilde{a}_k = h_k/\tilde{M}_k$. Using $z = x + iy$ as a parameter, one gets

$$a_k \leqslant \int |\varphi(z)|^{1/2}\, dx,$$

where the integration is along any complete horizontal line of $\tilde{S}_k$.

Applying the Schwarz inequality, we obtain

$$\begin{aligned}
a_k h_k &\leqslant \int\!\!\int_{\tilde{S}_k} |\varphi(z)|^{1/2}\, dx\, dy,\\
\sum a_k h_k &\leqslant \sum \int\!\!\int_{\tilde{S}_k} |\varphi(z)|^{1/2}\, dx\, dy = \int\!\!\int_{\cup \tilde{S}_k} |\varphi(z)|^{1/2}\, dx\, dy,\\
\Big(\sum a_k h_k\Big)^2 &\leqslant \Big(\sum \tilde{a}_k h_k\Big) \int\!\!\int_{\cup \tilde{S}_k} |\varphi(z)|\, dx\, dy\\
&\leqslant \Big(\sum \tilde{a}_k h_k\Big) \int\!\!\int_{S} |\varphi(z)|\, dx\, dy\\
&= \Big(\sum \tilde{a}_k h_k\Big) \Big(\sum a_k h_k\Big).
\end{aligned}$$

This proves that

$$\sum a_k h_k = \sum \frac{h_k^2}{M_k} \leqslant \sum \frac{h_k^2}{\tilde{M}_k}.$$

If the equality holds, then all the inequalities must be equalities. Therefore $\tilde{R}_k$ is a subannulus of R_k, swept out by closed trajectories, and $\cup \tilde{S}_k = S$; thus $\tilde{R}_k = R_k$. $\square$

Remark 2.5. Lemma 2.4 is the starting point of the proof of Theorem 1.6 given in [**29**].

Proof of Theorem 1.6. From $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{3g-3-p}) \to 0+$, we see that $\varepsilon_i \leqslant \varepsilon_i(0)$ for some $\varepsilon_0 = (\varepsilon_1(0), \ldots, \varepsilon_{3g-3-p}(0))$. Theorem 2.3 shows there exists a Jenkins–Strebel differential $\varphi_{\varepsilon_0} \in Q(S)$ with type Γ, and its characteristic cylinders $\{R_k\}$ have heights $(h_1, \ldots, h_p, \varepsilon_1(0), \ldots, \varepsilon_{3g-3-p}(0))$. Assuming that the characteristic rings of φ_{ε_0} have moduli $\{M_k\}$, and that the characteristic cylinders of φ_ε have moduli $\{M_k^\varepsilon\}$, by applying Lemma 2.4 we obtain

$$\begin{aligned}\|\varphi_\varepsilon\| &= \sum_{k=1}^{p} \frac{h_k^2}{M_k^\varepsilon} + \sum_{k=1}^{3g-3-p} \frac{\varepsilon_k^2}{M_k^\varepsilon} \\ &\leqslant \sum_{k=1}^{p} \frac{h_k^2}{M_k} + \sum_{k=1}^{3g-3-p} \frac{\varepsilon_k^2}{M_k} \\ &\leqslant \sum_{k=1}^{p} \frac{h_k^2}{M_k} + \sum_{k=1}^{3g-3-p} \frac{\varepsilon_k(0)^2}{M_k} \\ &= \|\varphi_{\varepsilon_0}\|.\end{aligned}$$

The uniqueness part of Theorem 1.6 follows immediately from Lemma 2.4. Thus the proof of Theorem 1.6 is complete. □

3. *Proof of Theorem* 2.3

Both the spaces $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ and $T(S_0)$ are homeomorphic to the $(6g-6)$-dimensional Euclidean space $\mathbb{R}^{6g-6}$. In order to prove the assertion of Theorem 2.3 (that is, that the mapping $h_H : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \longrightarrow T(S_0)$ is a homeomorphism), it is sufficient to check the following three assertions by Brouwer's theorem on the invariance of a domain:

(1) h_H is continuous;
(2) h_H is injective;
(3) h_H is proper.

Let $\{v_n\}_{n=1,2,\dots} \subset \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ be a sequence satisfying $v_n \to v_0 \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$. As a notational abbreviation, we write

$$v_n = (l_n, \theta_n) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}, \qquad \text{for } n = 0, 1, \dots.$$

Then $l_n \to l_0$, $\theta_n \to \theta_0$.

Denote $Q \longrightarrow T(S_0)$ the fibre bundle, whose fibre above a point $[S_\sigma] \in T(S_0)$ is the quadratic differential space $Q(S_\sigma)$. Let Σ be the space of free homotopy classes of simple closed curves on S_0, and let $\mathbb{R}^\Sigma$ be the functional space of Σ. Then as an infinite-dimensional linear space, the space $\mathbb{R}^\Sigma$ inherits the product topology.

In the product topology, Hubbard and Masur [**12**] showed that the mapping

$$\begin{gathered} p : Q \longrightarrow T(S_0) \times \mathbb{R}^\Sigma, \\ p((S_\sigma, \varphi)) = (S_\sigma, \gamma \to h_\varphi(\gamma)), \end{gathered}$$

is continuous.

To each $[h_H(v_n)] \in T(S_0)$, let $\varphi_H^{(n)}$ be the quadratic differential on $h_H(v_n)$ with type Γ and heights $\{h_k\}$. (When there is no confusion, the superscript (n) is omitted.) From the explicit construction of quasi-conformal homeomorphisms, we can easily deduce that the set $\{\tau_n = (h_H(v_n), \varphi_H)\}(n = 1, 2, \dots)$ lies in a compact set of the space Q. By passing to a subsequence, we may assume that $\tau_{n_k} \to \tau_0 = (S_\sigma, \varphi_0) \in Q$ as $k \to +\infty$. Since all the quadratic differentials φ_H on $h_H(v_n)$ have type Γ and heights $\{h_k\}$, where $n = 1, \dots$, the continuity of $p : Q \longrightarrow T(S_0) \times \mathbb{R}^\Sigma$ and, [**12**, Lemma 2.10] show that the quadratic differential $\varphi_0 \in Q(S_\sigma)$ also has type Γ, and its characteristic cylinders have heights $\{h_k\}$.

Similarly, $\tau'_{n_k} = (h_H(v_{n_k}), -\varphi_H) \to \tau'_0 = (S_\sigma, -\varphi_0)$ as $k \to \infty$. Thus we have $\tau_0 = (h_H(v_0), \varphi_H)$. This immediately implies that $S_\sigma = h_H(v_0)$. By a standard argument, we conclude that $\tau_n \to \tau_0$ as $n \to \infty$. The above argument shows that $h_H(v_n) \to h_H(v_0)$, which proves assertion (1).

Assertion (2) is a consequence of the monotonicity condition in Lemma 2.4.

In order to prove assertion (3), we shall show that if any sequence $\{v_n\}_{n=1,2\dots} \subset \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ approaches the boundary of $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, then $\{h_H(v_n)\}$ approaches the boundary of $T(S_0)$. Roughly speaking, the assertion that v_n approaches the boundary of $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ is equivalent to one or more of the following statements.

(i) As $n \to +\infty$, $l_k^{(n)} \to +\infty$ for some fixed k, $1 \leqslant k \leqslant 3g-3$.

(ii) As $n \to +\infty$, $l_k^{(n)} \to +0$ for some fixed k, $1 \leqslant k \leqslant 3g-3$.

(iii) As $n \to +\infty$,

$$c < l_k^{(n)} < C \quad \text{and} \quad \sum_{k=1}^{3g-3} \left|\theta_k^{(n)}\right| \to +\infty,$$

where $c, C > 0$ are two positive constants independent of n.

Given the vector $v_* = (1, 1, \dots, 1, 0, 0, \dots, 0) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, we obtain the Riemann surface $h_H(v_*) \in T(S_0)$ with the quadratic differential φ_H^* of type Γ, and its characteristic cylinders $\{R_k^*\}$ have heights $\{h_k\}$. Denote the moduli of $\{R_k^*\}$ by $\{M_k^*\}$.

Let $f_n : h_H(v_*) \longrightarrow h_H(v_n)$ be the extremal quasi-conformal homeomorphism homotopic to the identity, with maximal dilatation $K_{f_n} = K_n$. Supposing that $\tilde{R}_k^{(n)} = f_n(R_k^*)$ is the image ring domain on $h_H(v_n)$ and denoting the modulus of $\tilde{R}_k^{(n)}$ by $\tilde{M}_k^{(n)}$, we have $1/\tilde{M}_k^{(n)} \leqslant K_n / M_k^*$. Summing this inequality over all characteristic cylinders on the surface $h_H(v_*)$, one obtains

$$\sum \frac{h_k^2}{\tilde{M}_k^{(n)}} \leqslant K_n \sum \frac{h_k^2}{M_k^*}.$$

Let $M_k^{(n)}$ be the modulus of the characteristic cylinder of $\varphi_H^{(n)}$ on $h_H(v_n)$. It is immediate from Lemma 2.4 that

$$\sum \frac{h_k^2}{M_k^{(n)}} \leqslant \sum \frac{h_k^2}{\tilde{M}_k^{(n)}} \leqslant K_n \sum \frac{h_k^2}{M_K^*}.$$

That is,

$$\sum \left(h_k \cdot l_k^{(n)}\right) \leqslant K_n \sum (h_k \cdot 1).$$

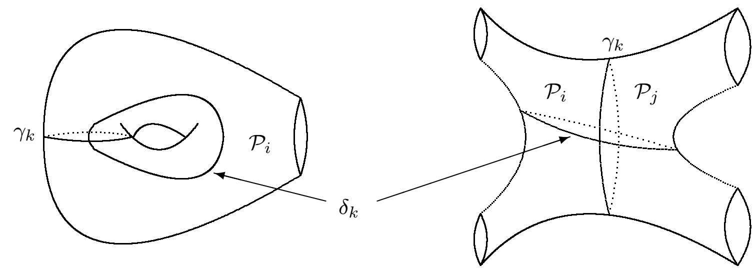
The left- and right-hand sides of the above inequality are the norms of $\varphi_H^{(n)}$ and φ_H^* respectively. In case (i), $l_k^{(n)} \to +\infty$ as $n \to +\infty$; hence $K_n \to +\infty$. Thus

$$d_T(h_H(v_*), h_H(v_n)) = \log K_n \to +\infty,$$

which proves case (i).

Let $i : R_k^{(n)} \longrightarrow h_H(v_n)$ be the embedding mapping. In the complete hyperbolic metric ρ on $R_k^{(n)}$, the ρ-length of the geodesic in homotopy class $[\gamma_k]$ is

$$l_\rho([\gamma_k]) = \frac{\pi}{M_k^{(n)}} = \frac{\pi \cdot l_k^{(n)}}{h_k} \to +0.$$



(a) (b)

FIGURE 2.

Ahlfors' lemma (see [**2**]) on decreasing hyperbolic metrics between $R_k^{(n)}$ and $h_H(v_n)$ and Wolpert's lemma [**1**, § 2.2, Theorem 4]) show that $d_{\mathrm{T}}(h_H(v_*), h_H(v_n)) \to +\infty$ as $n \to \infty$, which implies that case (ii) holds.

To prove case (iii), the following result is needed.

Denote by $M(S_0)$ the mapping class group of S_0. $M(S_0)$ serves as the modular group acting on the Teichmüller space $T(S_0)$. If $N = (n_1, n_2, \dots, n_{3g-3})$ is an integer array, we define $[f_N] \in M(S_0)$ as an element of $M(S_0)$ by applying n_k times Dehn twists with respect to γ_k for $1 \leqslant k \leqslant 3g-3$.

LEMMA 3.1. *Two integer arrays $N = \tilde{N}$ if and only if $[f_N] = [f_{\tilde{N}}]$ in the group $M(S_0)$.*

Proof. If $N \neq \tilde{N}$, then $n_k \neq \tilde{n}_k$ for some $k \in \{1, 2, \dots, 3g-3\}$.

From the pair $c_k = (c_{i\mu}, c_{jv})$ in the cubic graph $\mathcal{G}_\Gamma$, we set $\mathcal{X}_k = \mathcal{P}_i \cup \gamma_k \cup \mathcal{P}_j$ and choose the simple closed curve δ_k as in Figures 2(a) and (b). (Figure 2(a) corresponds to $\mathcal{P}_i = \mathcal{P}_j$ and Figure 2(b) corresponds to $\mathcal{P}_i \neq \mathcal{P}_j$.)

Obviously, the homotopy classes $[\delta_k] \cap [\gamma_i] = \varnothing$ if and only if $i \neq k$. Because $[\gamma_k]$ is not trivial, then $[f_N]([\delta_k]) \neq [f_{\tilde{N}}]([\delta_k])$; thus $[f_N] \neq [f_{\tilde{N}}]$ in M(S_0).

The lemma is proved. $\square$

Now let us proceed with the proof of Theorem 2.3.

For each n, by setting $\tilde{v}_n = (1, 1, \dots, 1, [\theta_1^{(n)}], [\theta_2^{(n)}], \dots, [\theta_{3g-3}^{(n)}])$, where $[\theta_k^{(n)}]$ is the Gauss sign, we obtain

$$d_{\mathrm{T}}(h_H(v_*), h_H(v_n)) \geqslant d_{\mathrm{T}}(h_H(v_*), h_H(\tilde{v}_n)) - d_{\mathrm{T}}(h_H(\tilde{v}_n), h_H(v_n)).$$

The facts that $c < l_k^{(n)} < C$ and $0 \leqslant \theta_k^{(n)} - [\theta_k^{(n)}] < 1$ imply that $d_{\mathrm{T}}(h_H(\tilde{v}_n), h_H(v_n)) \leqslant M$ for some constant $M > 0$ independent of n. Hence

$$d_{\mathrm{T}}(h_H(v_*), h_H(v_n)) \geqslant d_{\mathrm{T}}(h_H(v_*), h_H(\tilde{v}_n)) - M.$$

To avoid notational complication, we suppose that all $\theta_k^{(n)}$ are integers; then

$$\tilde{v}_n = \bigl(1, \dots, 1, \theta_1^{(n)}, \dots, \theta_{3g-3}^{(n)}\bigr).$$

Let $\theta_n = (\theta_1^{(n)}, \dots, \theta_{3g-3}^{(n)})$, where $n = 1, 2, \dots$. From the definition, $h_H(\tilde{v}_n) = [f_{\theta_n}](h_H(v_*))$ as points in the Teichmüller space $T(S_0)$. Since $\sum_{k=1}^{3g-3} |\theta_k^{(n)}| \to +\infty$, without loss of generality, we assume that $\theta_n \neq \theta_m$ if $n \neq m$. Lemma 3.1 shows that $[f_{\theta_n}] \neq [f_{\theta_m}]$. Therefore

$$d_{\mathrm{T}}(h_H(v_*), h_H(\tilde{v}_n)) = d_{\mathrm{T}}\big(h_H(v_*), \big[f_{\theta_n}\big](h_H(v_*)\big) \to +\infty$$

follows from the discreteness of Teichmüller modular group $M(S_0)$ acting on the Teichmüller space $T(S_0)$. Hence

$$d_{\mathrm{T}}(h_H(v_*), h_H(v_n)) \geqslant d_{\mathrm{T}}(h_H(v_*), h_H(\tilde{v}_n)) - M \to +\infty,$$

which proves assertion (3).

Therefore the proof of the 'height problem' is complete. □

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