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PARABOLIC RECTANGLE PACKINGS

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ABSTRACT

The **contacts graph** (or **nerve**) of a packing is a combinatorial graph which describes the combinatorics of the packing. Let G be the 1-skeleton of a triangulation of an open disk and let P be a **rectangle packing** with contact graph G. In this paper a topological criterion for deciding whether G is an α-EL parabolic graph is given. Our result shows the internal relation between the topological property of the packing P and the combinatorial property of the contacts graph G of P.

1. Introduction

A **packing** is a collection of compact connected sets with disjoint interiors in the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$. In this paper we shall consider rectangle packings with edges parallel to the coordinate axes in $\mathbb{C}$.

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Given an indexed packing $P = (P_v : v \in V)$, one can define its **contacts graph** (or **nerve**) $G = G(P)$ as follows. The set of vertices of G is V, the indexing set for P, and an edge $\langle u, v \rangle$ appears in G precisely when the sets P_u and P_v intersect. Thus G encodes some of the combinatorics of P.

The circle packing theorem [12] says that for any finite planar graph G there is some packing of geometric circles in the plane with contacts graph G. This fantastic theorem has received much attention since Thurston conjectured that the Riemann map from a simply connected domain to the unit disk can be approximated by using circle packings with the prescribed nerves. The conjecture was later proved by Rodin and Sullivan [17]. Some proofs of the circle packing theorem appear in [1], [2], [6], [22], [16], [14], [18], [21], [11].

The circle packing is well studied. Therefore, it seems to be of some interest to investigate other special cases. In [19], according to Thurston's suggestions, O. Schramm investigated the case where the packing sets are rectangles. He showed how the squares tile a rectangle with the special combinatorial structure. What makes the case of rectangles especially interesting? If P is a packing of rectangles with edges parallel to the coordinate axes, and if the contacts graph of P is a triangulation of an open topological disk (that is, the 1-skeleton of a triangulation of an open topological disk), then the packing is actually a tiling. This follows from the following easy observation which shows that there will be no "gaps" between the rectangles.

Observation 1.1: *Let R_a, R_b, R_c be three rectangles whose edges are parallel to the coordinate axes. Suppose that the intersection of each pair of the rectangles is nonempty. Then $R_a \cap R_b \cap R_c \neq \emptyset$.*

In order to state other results, we introduce the notion of fat sets. Heuristically, a set is fat if its area is roughly proportional to the square of its diameter. This property also holds locally.

Here is the precise definition.

Definition 1.2: The open disk with center x and radius r will be denoted $D(x, r)$. Let $\tau > 0$. A measurable set $X \subset \hat{\mathbb{C}}$ is τ-**fat**, if for every $x \in X$, $x \neq \infty$, and for every $r > 0$ such that $D(x, r)$ does not contain X, the inequality $\operatorname{area}(X \cap D(x, r)) \geq \tau \operatorname{area}(D(x, r))$ holds.

In [7], Z. X. He and O. Schramm gave a type characterization theorem for those infinite fat packing $P = (P_v : v \in V)$, where each P_v is a smooth disk and a τ-fat set. Here the real number τ is independent of the vertex v.

Therefore, there is a natural problem about the packings of rectangles: Can we give a similar type characterization result for the packings of rectangles?

Suppose that $P = (P_v : v \in V)$ is a rectangle packing. Though each rectangle is a τ_v-fat set, there is no fixed number $\tau > 0$ such that P is a τ-fat packing. At the same time a rectangle is not a smooth disk.

In order to give an answer to the above question, we shall introduce the notions of an α-EL parabolic graph and an α^*-EL parabolic graph. α-EL (or α^*-EL) parabolicity is a combinatorial property. It is defined by using Cannon's vertex extremal length [5]. The precise definitions will appear later.

By using these notions, we shall prove

Theorem 1.3 (Type Characterization Theorem): *Let $P = \{P_v : v \in V\}$ be a rectangle packing in $\mathbb{C}$ whose edges are parallel to the coordinate axes. And let $\alpha : V \to (0, 1]$ be an assignment of weights to the vertices with $\alpha(v) = H_v/W_v$, where H_v is the height of the rectangle P_v and W_v is its width with $H_v \leq W_v$. Denote by $G = (V, E)$ the contacts graph of P. Assume that G is locally finite and connected.*

(1) *If the rectangle packing P is locally finite in $\mathbb{C}$, then G is α-EL parabolic.*
(2) *Conversely, suppose the contact graph G is a triangulation of an open topological disk. If it is α^*-EL parabolic, then P is locally finite in $\mathbb{C}$.*

In the Appendix we will give an example of an α-EL parabolic packing that is not locally finite in the complex plane $\mathbb{C}$.

Remark: In this paper we say a packing $P = \{P_v : v \in V\}$ is locally finite in $\mathbb{C}$ if, for every compact subset K of $\mathbb{C}$,

$$P_v \cap K = \emptyset,$$

except for a finite number of P_v in P. We say a graph $G = (V, E)$ is locally finite if, for every vertex $v \in V$, the degree of v is finite. Here the degree of a vertex $v \in V$ is the number of edges it emanates.

This paper is organized as follows. In Section 2 we introduce the notion of discrete extremal length and give some basic properties of discrete extremal length. In Section 3 we give some elemental properties of fat sets and give the proof of the first part of Theorem 1.3. Some topological lemmas on rectangle packings are introduced in Section 4. The object of Section 5 is to show the connection between the α-Extremal Length and the α^*-Extremal Length. The proof of the second part of Theorem 1.3 was left to Section 6. In the Appendix we construct an example of an α-EL parabolic packing that is not locally finite in $\mathbb{C}$.

Notational Conventions: Throughout the paper, for any set A we denote by $|A|$ the cardinality of A. We will denote by $G = (V, E)$ a locally finite, infinite, connected graph, where $E = E(G)$ is the set of edges in G and $V = V(G)$ is the set of its vertices.

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2. Discrete extremal length

In this section we shall define the general **discrete extremal length** in an infinite graph. The extremal length of a set of paths in a graph is the discrete counterpart of the extremal length of a family of curves in a Riemannian manifold. It was first introduced by Duffin (1962) for finite graphs and was subsequently studied in the infinite case by J. Cannon, Z. X. He, O. Schramm and others (see, e.g., [5], [7]).

We present here the basic definitions and properties which will be used in the sequel. Let $G = (V, E)$ be a locally finite connected graph. It will always be a simple graph; that is, each edge has two distinct vertices, and there is at most one edge joining any two vertices.

A **path** $\gamma \subset G$ is a finite or infinite sequence $(v_0, v_1, \ldots)$ of vertices such that $\langle v_i, v_{i+1}\rangle \in E$ for $i = 0, 1, \ldots$. Denote by $V(\gamma) = \{v_0, v_1, \ldots\}$ the vertices of γ. For convenience we write $v \in \gamma$ instead of $v \in V(\gamma)$. Similarly, for any set of paths $\Gamma = \{\gamma\}$ in the graph G, we set $V(\Gamma) = \{V(\gamma) : \gamma \in \Gamma\}$. A set $A \subset V$ of vertices is said to be **connected**, if for every $v, w \in A$, there is a path γ in G from v to w with $V(\gamma) \subset A$. (We allow trivial paths which contain only one vertex.)

Let a function $\alpha : V \to (0,1]$ be an assignment of weights to the vertices. A nonnegative function $m : V \to [0,\infty)$ is called a (discrete) **metric** on G. Given a path γ and a metric m, we define the m-**length** and the m-**dual-length** of γ, respectively, to be

$$\text{Length}_m(\gamma) = \sum_{v\in\gamma} m(v);$$

$$\text{Dual-Length}_m(\gamma) = \sum_{v\in\gamma} \alpha(v)m(v).$$

Note that a shortest path from a vertex v_0 to itself is the path $\gamma_0 = (v_0)$, and its length (or dual-length) is $\text{Length}_m(\gamma_0) = m(v_0)$ (or $\text{Dual-Length}_m(\gamma_0) = \alpha(v_0)m(v_0)$).

If Γ is a collection of paths of G, then we define its m-**length** (or m-**dual-length**) to be the least m-length (or m-dual-length) of a path in Γ:

$$\text{Length}_m(\Gamma) = \inf_{\gamma\in\Gamma}\{\,\text{Length}_m(\gamma)\,\}$$

$$(\text{or}\quad \text{Dual-Length}_m(\Gamma) = \inf_{\gamma\in\Gamma}\{\,\text{Dual-Length}_m(\gamma)\,\}).$$

For any metric $m : V \to [0,\infty)$ on G, we define its α-area $\| m \|_\alpha^2$ by

$$\text{area}_\alpha(m) = \| m \|_\alpha^2 = \sum_{v\in V} \alpha(v)\cdot m(v)^2.$$

The collection of all metrics m on G with $0 < \text{area}_\alpha(m) < \infty$ will be denoted by $\mathcal{M}_\alpha(V)$.

Finally, the α-**extremal length** and the α^*-**extremal length** of Γ are, respectively, defined as

$$\text{EL}_\alpha(\Gamma) = \sup\left\{ \frac{\big(\text{Length}_m(\Gamma)\big)^2}{\text{area}_\alpha(m)} : m \in \mathcal{M}_\alpha(V) \right\};$$

$$\text{EL}_\alpha^*(\Gamma) = \sup\left\{ \frac{\big(\text{Dual-Length}_m(\Gamma)\big)^2}{\text{area}_\alpha(m)} : m \in \mathcal{M}_\alpha(V) \right\}.$$

These are two numbers in $[0,\infty]$. Note that the ratio $\big(\text{Length}_m(\Gamma)\big)^2/\text{area}_\alpha(m)$ (or $\big(\text{Dual-Length}_m(\Gamma)\big)^2/\text{area}_\alpha(m)$) is independent of a positive constant multiple of the metric m.

Given subsets $A, B \subset V$, we denote by $\Gamma(A,B) = \Gamma_G(A,B)$ the set of all paths in G with initial point in A and terminal point in B. The α-**extremal**

length EL_α and α^***-extremal length** EL_α between A and B are, respectively, defined by

$$\mathrm{EL}_\alpha = \mathrm{EL}_\alpha(A,B) = \mathrm{EL}_\alpha(\Gamma(A,B));$$

$$\mathrm{EL}^*_\alpha = \mathrm{EL}^*_\alpha(A,B) = \mathrm{EL}^*_\alpha(\Gamma(A,B)).$$

An infinite path γ in G is **transient** if it contains infinitely many distinct vertices. The set of transient paths in G with the initial point in A will be denoted by $\Gamma(A,\infty)$. The α-**extremal length** and α^*-**extremal length** from A to ∞ are, respectively, defined as

$$\mathrm{EL}_\alpha(A,\infty) = \mathrm{EL}_\alpha(\Gamma(A,\infty));$$

$$\mathrm{EL}^*_\alpha(A,\infty) = \mathrm{EL}^*_\alpha(\Gamma(A,\infty)).$$

To make the definitions of $\mathrm{EL}_\alpha(A,\infty)$ and $\mathrm{EL}^*_\alpha(A,\infty)$ more explicit, we have

$$\begin{aligned}\mathrm{EL}_\alpha(A,\infty) &= \sup_m \inf_\gamma \left\{ \frac{\big(\mathrm{Length}_m(\gamma)\big)^2}{\mathrm{area}_\alpha(m)} \right\} \\ &= \sup_m \inf_\gamma \left\{ \frac{\Big(\sum_{v\in\gamma} m(v)\Big)^2}{\sum_{v\in V} \alpha(v)\cdot m(v)^2} \right\};\\ \mathrm{EL}^*_\alpha(A,\infty) &= \sup_m \inf_\gamma \left\{ \frac{\big(\text{Dual-Length}_m(\gamma)\big)^2}{\mathrm{area}_\alpha(m)} \right\} \\ &= \sup_m \inf_\gamma \left\{ \frac{\Big(\sum_{v\in\gamma} \alpha(v)m(v)\Big)^2}{\sum_{v\in V} \alpha(v)\cdot m(v)^2} \right\}.\end{aligned}$$

Here m runs over $\mathcal{M}_\alpha(V)$ and γ runs over $\Gamma_G(A,\infty)$. Of course, these make sense only for an infinite graph G.

An infinite graph G is α-EL **parabolic** (or α^*-EL **parabolic**) if $\mathrm{EL}_\alpha(\{v\},\infty) = \infty$ (or $\mathrm{EL}^*_\alpha(\{v\},\infty) = \infty$) for some $v \in V$. Otherwise, G is α-EL **hyperbolic** (or α^*-EL **hyperbolic**).

For a metric m, we let $d^*_m(A,B)$ (respectively, $d^*_m(A,\infty)$) denote the dual-distance from A to B(respectively, from A to ∞) in the metric m. That is,

$$d^*_m(A,B) = \text{Dual-Length}_m(\Gamma(A,B)) = \inf\{\text{Dual-Length}_m(\gamma) : \gamma \in \Gamma(A,B)\},$$

$$d^*_m(A,\infty) = \text{Dual-Length}_m(\Gamma(A,\infty)) = \inf\{\text{Dual-Length}_m(\gamma) : \gamma \in \Gamma(A,\infty)\}.$$

These definitions give a discrete analog for the classical notion of extremal length ([13] is a good introduction to continuous extremal length). The vertex extremal length was introduced by J. Cannon [5]. Cannon's motivation was to obtain criteria for deciding whether a group acts conformally on the Riemann sphere $\hat{\mathbb{C}}$. Later, Z. He and O. Schramm [7] discovered that the extremal metrics of vertex extremal length give square tilings of rectangles with prescribed contacts. That is, these metrics realize the supremum in the definition of the extremal length.

At the end of this section, we give an elementary combinatorial of the infinite graph, which will be needed below.

Proposition 2.1: *Let $G = (V, E)$ be an infinite graph and $v_0 \in V$ be a vertex. Let a function $\alpha : V \to (0, 1]$ be an assignment of weights to the vertices and let $\beta : V \to [0, \infty)$ be a non-negative function of the vertices. Let $\mathcal{M} = \mathcal{M}_\alpha(V)$ and $\Gamma = \Gamma(\{v_0\}, \infty)$. Then*

$$\sup_{m\in\mathcal{M}} \inf_{\gamma\in\Gamma} \left\{ \frac{\left(\sum_{v\in\gamma} \beta(v)m(v)\right)^2}{\sum_{v\in V} \alpha(v)\cdot m(v)^2} \right\} = \infty$$

if and only if there exists a finite α-area metric $m_0 \in \mathcal{M}_\alpha(V)$ such that

$$\inf \left\{ \sum_{v\in V} \beta(v)m_0(v) : \gamma \in \Gamma \right\} = \infty.$$

Proof. The sufficiency of this proposition is obvious, so we only need to prove the necessity. In what follows, for notational convenience we denote

$$d_m^\beta \equiv \inf \left\{ \sum_{v\in V} \beta(v)m(v) : \gamma \in \Gamma \right\},$$

where $m \in \mathcal{M}_\alpha(V)$ is any metric.

Now we prove the necessity. Suppose that

$$\sup_{m\in\mathcal{M}} \inf_{\gamma\in\Gamma} \left\{ \frac{\left(\sum_{v\in\gamma} \beta(v)m(v)\right)^2}{\sum_{v\in V} \alpha(v)\cdot m(v)^2} \right\} = \infty.$$

Then, for any $j \in \mathbb{N}$, there exists a metric $\tilde{m}_j \in \mathcal{M}_\alpha(V)$ such that

$$\frac{(d_{\tilde{m}_j}^\beta)^2}{\| \tilde{m}_j \|_\alpha^2} > 2^j. \tag{2.1}$$

If $d^{\beta}_{\tilde{m}_j} = \infty$ for some j, then the conclusion obviously holds. In fact $\tilde{m}_j$ is the metric we need. So we assume without loss of generality that $d^{\beta}_{\tilde{m}_j} < \infty$ for every $j \in \mathbb{N}$. Since $\tilde{m}_j \in \mathcal{M}_\alpha(V)$ (that is $0 <\| \tilde{m}_j \|_\alpha < \infty$), and $d^{\beta}_{\tilde{m}_j}$ satisfies the inequality (2.1), we have $0 < d^{\beta}_{\tilde{m}_j} < \infty$. Define a series of new metrics as follows:

$$m_j(v) = \frac{1}{d^{\beta}_{\tilde{m}_j}} \cdot \tilde{m}_j(v) \quad \text{for each } v \in V.$$

Note that $d^{\beta}_{c\,\tilde{m}_j} = c \cdot d^{\beta}_{\tilde{m}_j}$ when c is a positive constant. Therefore, we have $d^{\beta}_{m_j} = 1$ and

$$\frac{1}{\| m_j \|^2_\alpha} = \frac{(d^{\beta}_{m_j})^2}{\| m_j \|^2_\alpha} = \frac{(d^{\beta}_{\tilde{m}_j})^2}{\| \tilde{m}_j \|^2_\alpha} > 2^j.$$

Hence

$$\| m_j \|^2_\alpha = \sum_{v \in V} \alpha(v) m_j^2(v) < 2^{-j}. \tag{2.2}$$

Define a metric m_0 on G by setting

$$m_0(v) = \sum_{j=1}^{\infty} \frac{m_j(v)}{j} \quad \text{for each } v \in V.$$

It follows from (2.2) that, for each $v \in V$,

$$\begin{aligned} \alpha(v) m_0^2(v) &= \Big(\sum_{j=1}^{\infty} \sqrt{\alpha(v)} \frac{m_j(v)}{j} \Big)^2 \\ &\le \Big(\sum_{j=1}^{\infty} \frac{1}{j^2} \Big) \Big(\sum_{j=1}^{\infty} \alpha(v) m_j^2(v) \Big) \\ &\le \Big(\sum_{j=1}^{\infty} \frac{1}{j^2} \Big) \Big(\sum_{j=1}^{\infty} \| m_j \|^2_\alpha \Big) \\ &< \Big(\sum_{j=1}^{\infty} \frac{1}{j^2} \Big) \Big(\sum_{j=1}^{\infty} 2^{-j} \Big) \\ &< \infty. \end{aligned} \tag{2.3}$$

Since $\alpha(v) > 0$, the inequality (2.3) shows that $0 \le m_0(v) < \infty$ for each $v \in V$. This implies that the new metric m_0 is well defined. Using the Schwartz inequality and (2.2), we get an estimate for the α-area of m_0 as follows:

$$
\begin{aligned}
\operatorname{area}_\alpha(m_0) &= \sum_{v\in V} \alpha(v) m_0^2(v) \\
&= \sum_{v\in V} \left(\sum_{j=1}^{\infty} \frac{\sqrt{\alpha(v)}m_j(v)}{j} \right)^2 \\
&\le \sum_{v\in V} \left\{ \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) \left(\sum_{j=1}^{\infty} \alpha(v) m_j^2(v) \right) \right\} \\
&= \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) \left(\sum_{j=1}^{\infty} \sum_{v\in V} \alpha(v) m_j^2(v) \right) \\
&< \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) \left(\sum_{j=1}^{\infty} 2^{-j} \right) \\
&< C_1,
\end{aligned}
\tag{2.4}
$$

where $C_1 > 0$ is a universal constant. Noting $\|m_0\|_\alpha^2 \ge \|\tilde{m}_1\|_\alpha^2/(d_{\tilde{m}_1}^\beta)^2 > 0$, thus $m_0 \in \mathcal{M}_\alpha(V)$.

Since $d_{m_j}^\beta = 1$, we have

$$\sum_{v\in\gamma} \beta(v) m_j(v) \ge 1$$

for every $\gamma \in \Gamma$. So for each $\gamma \in \Gamma$ it follows that

$$\sum_{v\in\gamma} \frac{\beta(v) m_j(v)}{j} \ge \frac{1}{j}.$$

Therefore, for each $\gamma \in \Gamma$, we have

$$\sum_{v\in\gamma} \beta(v) m_0(v) = \sum_{j=1}^{\infty} \sum_{v\in\gamma} \frac{\beta(v) m_j(v)}{j} = \infty.$$

Note $m_0 \in \mathcal{M}_\alpha(V)$. We get $d_{m_0}^\beta = \infty$. So we complete the proof of Proposition 2.1. ∎

3. Some geometric behavior of rectangle packing

In Section 1 we give the definition of a fat set. Here we will show some basic properties of such sets.

Fact 3.1: *Let F be a τ-fat set, $\tau > 0$. Then, for every $z \in \mathbb{C}$ and $r > 0$,*

$$\text{area}(D(z,3r) \cap F) \geq \pi\tau \,\text{diameter}(D(z,r) \cap F)^2 \tag{3.1}$$

holds.

Proof. Let $x, y \in D(z,r) \cap F$. It is clear that $D(x, |y-x|) \subset D(z,3r)$. By the τ-fatness of F, we have

$$\text{area}(D(z,3r) \cap F) \geq \text{area}(D(x,|y-x|) \cap F) \geq \pi\tau|y-x|^2.$$

The fact follows. ∎

Lemma 3.2: *Let F be a connected τ-fat set in $\hat{\mathbb{C}}$ and let g be a Möbius transformation. Then $F^* = g(F)$ is a τ^*-fat set, where $\tau^* = \tau/200$.*

Remark: For the proof of Lemma 3.2 refer to [20].

Fact 3.3: *Let R be a rectangle whose edges are parallel to the coordinate axes. Suppose that h is the height of the rectangle and w its width with $h \leq w$. Let $k = h/w$. Then the rectangle R is a $k/(\pi(k^2+1))$-fat set.*

Proof. Let $R = [a,b] \times [c,d]$ be a rectangle with four vertices $z_i, i = 1, \ldots, 4$. See Figure 1. So R has height $h = d - c$ and width $w = b - a$.

Obviously, for any positive number $r > 0$, $D(z_1, r)$ does not contain R if and only if $0 < r \leq \sqrt{w^2+h^2}$. Hence, for $0 < r \leq \sqrt{w^2+h^2}$, we have a low bound of $\text{area}(D(z_1,r) \cap R)/\text{area}(D(z_1,r))$.

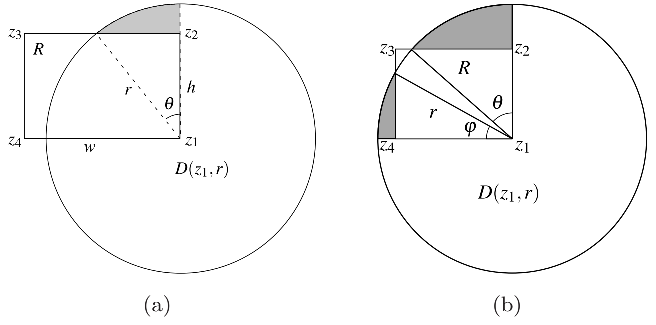

Figure 1

If $0 < r \leq h$, it is easy to see that

$$\frac{\text{area}(D(z_1, r) \cap R)}{\text{area}(D(z_1, r))} = 1/4.$$

If $h < r \leq w$, let $S = \Omega \cap D(z_1, r) - R$, where $\Omega = \{(x, y) \in \mathbb{R}^2 : x < b \text{ and } y > c\}$ (the shaded region in Figure 1 (a)). It is clear that

$$\begin{aligned}\text{area}(S) &= (1/2)\,(\theta r^2 - r^2 \sin\theta\cos\theta) \\ &= (r^2/4)(2\theta - \sin(2\theta)),\end{aligned}$$

where the angle θ is as in Figure 1 (a). Thus we have

$$\frac{\text{area}(S)}{\text{area}(D(z_1, r))} = (1/(4\pi))(2\theta - \sin(2\theta)).$$

Therefore, it follows that

$$\begin{aligned}\frac{\text{area}(D(z_1, r) \cap R)}{\text{area}(D(z_1, r))} &= \frac{\text{area}(D(z_1, r) \cap \Omega)}{\text{area}(D(z_1, r))} - \frac{\text{area}(S)}{\text{area}(D(z_1, r))} \\ &= 1/4 - (1/(4\pi))(2\theta - \sin(2\theta)).\end{aligned}$$

We know that $(1/(4\pi))(2\theta - \sin(2\theta))$ is an increasing function of $\theta \in (0, \pi/2)$. So $\text{area}(D(z_1, r) \cap R)/\text{area}(D(z_1, r))$ is a decreasing function of $r \in (h, w]$.

If $w < r \leq \sqrt{w^2 + h^2}$, let S_1 be the connected component of $\Omega \cap D(z_1, r) - R$ which contains the point $z_2 = (b, d)$, and let S_2 be the connected component of $\Omega \cap D(z_1, r) - R$ which contains the point $z_4 = (a, c)$ (the shaded regions, respectively, in Figure 1 (b)). It is clear that

$$\begin{aligned}\text{area}(S_1) &= (1/2)\,(\theta r^2 - r^2 \sin\theta\cos\theta) \\ &= (r^2/4)(2\theta - \sin(2\theta)); \\ \text{area}(S_2) &= (1/2)\,(\varphi r^2 - r^2 \sin\varphi\cos\varphi) \\ &= (r^2/4)(2\varphi - \sin(2\varphi)),\end{aligned}$$

where the angles θ and φ are as in Figure 1 (b). Thus we have

$$\frac{\text{area}(S_1) + \text{area}(S_2)}{\text{area}(D(z_1, r))} = (1/(4\pi))(2\theta - \sin(2\theta)) + (1/(4\pi))(2\varphi - \sin(2\varphi)).$$

Therefore,

$$\begin{aligned}\frac{\text{area}(D(z_1, r) \cap R)}{\text{area}(D(z_1, r))} &= \frac{\text{area}(D(z_1, r) \cap \Omega)}{\text{area}(D(z_1, r))} - \frac{\text{area}(S_1) + \text{area}(S_2)}{\text{area}(D(z_1, r))} \\ &= 1/4 - (1/(4\pi))(2\theta - \sin(2\theta)) - (1/(4\pi))(2\varphi - \sin(2\varphi)).\end{aligned}$$

By the same reasoning we deduce that $\mathrm{area}(D(z_1,r)\cap R)/\mathrm{area}(D(z_1,r))$ is a decreasing function of $r\in(w,\sqrt{h^2+w^2}]$.

So, for $0<r\le\sqrt{h^2+w^2}$, we get

$$\frac{\mathrm{area}(D(z_1,r)\cap R)}{\mathrm{area}(D(z_1,r))}=\begin{cases}1/4 & \text{for } 0<r\le h;\\ 1/4-(1/(4\pi))(2\theta-\sin(2\theta)) & \text{for } h<r\le w;\\ 1/4-(1/(4\pi))(2\theta-\sin(2\theta)) & \\ \qquad -(1/(4\pi))(2\varphi-\sin(2\varphi)) & \text{for } w<r\le\sqrt{w^2+h^2}.\end{cases}$$

By the above arguments, we conclude that the function

$$s(r)=\frac{\mathrm{area}(D(z_1,r)\cap R)}{\mathrm{area}(D(z_1,r))}$$

is a non-increasing function of $r\in(0,\sqrt{h^2+w^2}]$. Thus, for any $0<r\le\sqrt{h^2+w^2}$, we have

$$\begin{aligned}\frac{\mathrm{area}(D(z_1,r)\cap R)}{\mathrm{area}(D(z_1,r))}&\ge\frac{\mathrm{area}(D(z_1,\sqrt{h^2+w^2})\cap R)}{\mathrm{area}(D(z_1,\sqrt{h^2+w^2}))}\\&=\frac{w\cdot h}{\pi(h^2+w^2)}\\&=\frac{k}{\pi(k^2+1)},\end{aligned}$$

where $k=h/w$.

Let $z\in R$ be any point in the rectangle R and $r>0$ be any positive real number with $D(z,r)$ not containing R. Without loss of generality, we assume the vertex z_1 of the rectangle nearest to z. Then $D(z_1,r)$ does not contain R, and

$$\frac{\mathrm{area}(D(z,r)\cap R)}{\mathrm{area}(D(z,r))}\ge\frac{\mathrm{area}(D(z_1,r)\cap R)}{\mathrm{area}(D(z_1,r))}\ge\frac{k}{\pi(k^2+1)},$$

where $k=h/w$. It implies that R is a $\frac{k}{\pi(k^2+1)}$-fat set. We complete the proof of Fact 3.3. ∎

Lemma 3.4: *Let $P=\{P_v : v\in V\}$, $\alpha:V\to(0,1]$ and $G=(V,E)$ be defined as in Theorem 1.3. Suppose that the rectangle packing P is locally finite in $\mathbb{C}$. Let $K\subset\mathbb{C}$ be a compact set. For every $A\subset\mathbb{C}$, let $V(A)$ denote the set of vertices $v\in V$ such that P_v intersects A. Then*

$$\sup_W\{EL_\alpha(V(K),V(W)) : \mathbb{C}-W \textit{ is compact subset of } \mathbb{C}\}=\infty.$$

Proof. Denote by $C(z,r) = \partial D(z,r)$ the circle with center z and radius r. Since K is a compact set and the rectangle packing P is locally finite in $\mathbb{C}$, we deduce that the cardinality $|V(K)| < \infty$. Equivalently, the set $\bigcup_{v\in V(K)} P_v$ is a compact set. This implies that there is a positive real number r_1 such that the closure $\overline{D(0,r_1)} \supset \bigcup_{v\in V(K)} P_v$.

We define inductively a sequence of positive numbers $r_1 < r_2 < \cdots$.

The first number r_1 in this sequence has been defined already. We assume that, for some $n > 1$, the numbers $r_1, \ldots, r_{n-1}$ have been defined. We set

$$V^* \equiv \big\{ v \in V \;:\; P_v \cap \overline{D\big(0,\, 2r_{n-1}\big)} \neq \emptyset \big\}.$$

Since the packing P is locally finite in $\mathbb{C}$, we have $|V^*| < \infty$. Therefore $\bigcup_{v\in V^*} P_v$ is a compact set, which implies

$$\rho = \sup \Big\{ |z| \;:\; z \in \bigcup_{v\in V^*} P_v \Big\} < \infty. \tag{3.2}$$

Now we let r_n be sufficiently large so that $r_n > 2r_{n-1} + \rho$. From the choice of r_n and V^*, it follows from (3.2) that, for each vertex $v \in V$, either $P_v \cap C(0, r_n) = \emptyset$ or $P_v \cap C(0, 2r_{n-1}) = \emptyset$. That is,

$$V\big(C(0,2r_{n-1})\big) \cap V\big(C(0,r_n)\big) = \emptyset. \tag{3.3}$$

For each n let A_n be the closed annulus bounded by $C(0, r_n)$ and $C(0, 2r_n)$. Define a metric m on G by setting

$$m(v) = \sum_{n=1}^{\infty} \frac{\text{diameter}(P_v \cap A_n)}{n r_n},$$

for each $v \in V$. By the construction of the sequence r_n and (3.3), at most one term in this sum is nonzero.

Note that $\{P_v\}$ is a packing. From what we have just proved (Fact 3.1 and Fact 3.3), it follows that

$$
\begin{aligned}
\mathrm{area}_\alpha(m) &= \sum_{v\in V} \alpha(v) \left(\sum_{n=1}^{\infty} \frac{\mathrm{diameter}(P_v \cap A_n)}{n r_n} \right)^2 \\
&= \sum_{n=1}^{\infty} \sum_{v\in V} \frac{\mathrm{diameter}(P_v \cap A_n)^2 \cdot \alpha(v)}{n^2 r_n^2} \\
&\le \sum_{n=1}^{\infty} \sum_{v\in V} \frac{\mathrm{diameter}(P_v \cap D(0,2r_n))^2 \cdot \alpha(v)}{n^2 r_n^2} \\
&\le \sum_{n=1}^{\infty} \sum_{v\in V} \frac{\mathrm{area}(P_v \cap D(0,6r_n)) \cdot (\alpha(v)^2+1)}{n^2 r_n^2} \qquad (3.4) \\
&\le 2\sum_{n=1}^{\infty} \sum_{v\in V} \frac{\mathrm{area}(P_v \cap D(0,6r_n))}{n^2 r_n^2} \\
&\le 2\sum_{n=1}^{\infty} \frac{\mathrm{area}(D(0,6r_n))}{n^2 r_n^2} \\
&= 72\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&< C_2,
\end{aligned}
$$

where C_2 is a universal constant. So $m \in \mathcal{M}_\alpha(V)$.

Fix a positive integer N, and consider some path γ from $V(K)$ to $V\big(\mathbb{C} - \overline{D(0,r_N)}\big)$. For each integer $n \in [1, N-1]$ the union $\bigcup_{v\in V(\gamma)} P_v$ is a connected set that intersects the two circles $C(0,r_n)$ and $C(0,2r_n)$ which form the boundary of A_n. Therefore, for such n, $\sum_{v\in\gamma} \mathrm{diameter}(P_v \cap A_n) \ge r_n$. This then implies that

$$
\begin{aligned}
\mathrm{Length}_m(\gamma) &= \sum_{v\in\gamma} \sum_{n=1}^{\infty} \frac{\mathrm{diameter}(P_v \cap A_n)}{n r_n} \\
&\ge \sum_{n=1}^{N-1} \sum_{v\in\gamma} \frac{\mathrm{diameter}(P_v \cap A_n)}{n r_n} \qquad (3.5) \\
&\ge \sum_{n=1}^{N-1} \frac{1}{n},
\end{aligned}
$$

which tends to infinity as $N \to \infty$. Since $\text{area}_\alpha(m) < \infty$, we get

$$EL_\alpha\big(V(K), V(\mathbb{C} - \overline{D(0, r_N)})\big) \to \infty,$$

as $N \to \infty$, which proves the lemma. ∎

Proof of theorem 1.3 (1) . Pick some $v_0 \in V$. Denote $\Gamma = \Gamma(\{v_0\}, \infty)$. In what follows, we will prove that $\text{EL}_\alpha(\{v_0, \infty\}) = \infty$.

We assume, by contradiction, that there exists a finite positive number M such that

$$\text{(3.6)} \qquad \text{EL}_\alpha(\{v_0\}, \infty) = \sup_m \inf_\gamma \left\{ \frac{\big(\text{Length}_m(\gamma)\big)^2}{\text{area}_\alpha(m)} \right\} = M.$$

Here, m runs over $\mathcal{M}_\alpha(V)$ and γ runs over $\Gamma(\{v_0\}, \infty)$. From Lemma 3.4 with $K = P_{v_0}$, it follows that there exists an open set W with $\mathbb{C} - W$ being a compact such that

$$\text{(3.7)} \qquad \text{EL}_\alpha(V(K), V(W)) > 3M.$$

Hence there exists a metric $m_0 \in \mathcal{M}_\alpha(V)$ such that

$$\text{(3.8)} \qquad \frac{\big(\text{Length}_{m_0}(\gamma)\big)^2}{\text{area}_\alpha(m_0)} > 3M,$$

for every $\gamma \in \Gamma(K, W)$.

By equation (3.6), we know that

$$\inf_{\gamma \in \Gamma(\{v_0\}, \infty)} \frac{\big(\text{Length}_{m_0}(\gamma)\big)^2}{\text{area}_\alpha(m_0)} \le M.$$

Therefore, there exists a transient path $\widetilde{\gamma} \in \Gamma(\{v_0\}, \infty)$, $\widetilde{\gamma} = (u_0, u_1, \ldots, u_n, \ldots)$, such that

$$\text{(3.9)} \qquad \frac{\big(\text{Length}_{m_0}(\widetilde{\gamma})\big)^2}{\text{area}_\alpha(m_0)} < 2M.$$

We claim that there exists a vertex $u_j \in \widetilde{\gamma}$ such that $u_j \in V(W)$. That is, $P_{u_j} \cap W \neq \emptyset$. If each vertex $u_n \in \widetilde{\gamma}$ satisfies $u_n \notin V(W)$, then

$$P_{u_n} \subseteq \mathbb{C} - W.$$

Choose a point $z_n \in P^\circ_{u_n}$. Noting $\mathbb{C} - W$ is a compact set, without loss of generality, we assume that $z_n \to z_0$ for some point $z_0 \in \mathbb{C} - W$. Since $\widetilde{\gamma}$ is a transient path, we obtain that $\{P_{u_n}\}$ contains infinitely many distinct rectangles. So the point z_0 is an accumulation point of the packing $P = \{P_v : v \in V\}$, which contradicts the assumption that P is locally finite in $\mathbb{C}$.

Therefore, there exists a vertex $u_j \in \widetilde{\gamma}$ such that $u_j \in V(W)$. Let $\gamma^* = (u_0, \ldots, u_j)$. So $\gamma^* \in \Gamma(K, W)$ and $\text{Length}_{m_0}(\gamma^*) \leq \text{Length}_{m_0}(\widetilde{\gamma})$. By (3.8) and (3.9), we get

$$3M < \frac{\left(\text{Length}_{m_0}(\gamma^*)\right)^2}{\text{area}_\alpha(m_0)} \leq \frac{\left(\text{Length}_{m_0}(\widetilde{\gamma})\right)^2}{\text{area}_\alpha(m_0)} < 2M.$$

This contradiction implies that $\text{EL}_\alpha(\{v_0, \infty\})=\infty$. Thus G is α-EL parabolic. ∎

4. Topological behavior of rectangle packing

In this section we gather a few elementary topological lemmas which will be needed below. The reader is advised to skip the proofs at the first reading, and return to them later.

Recall that $P = \{P_v : v \in V\}$ is a packing of rectangles in the plane $\mathbb{C}$ and $G = (V; E)$ is the contacts graph of P.

Lemma 4.1: *Suppose that G is a disk triangulation graph. Then, for any rectangles $P_u \neq P_v$, we have either $P_u \cap P_v = \emptyset$ or $|P_u \cap P_v| = \infty$.*

Proof. Since the edges of every rectangle in P are parallel to the coordinate axes, it is clear that, for any $u, v \in V$, there are only three cases for $P_u \cap P_v$: (i) $P_u \cap P_v = \emptyset$; (ii) $|P_u \cap P_v| = 1$; (iii) $|P_u \cap P_v| = \infty$.

We assume, by contradiction, that $|P_u \cap P_v| = 1$. Denote $P_u \cap P_v = \{p\}$ (see Figure 2 (a)). Thus the edge $\langle u, v\rangle \in E$. By the definition of the triangulation of an open disk, we know that there are exactly two vertices w_1, w_2 such that $\langle u, v, w_1\rangle$ and $\langle u, v, w_2\rangle$ are two faces in G. This means $P_{w_1} \cap P_{w_2} = \{p\}$.

Since the graph G is the contacts graph of the packing P and $P_{w_1} \cap P_{w_2} \neq \emptyset$, there exists an edge $\langle w_1, w_2\rangle \in E$ connecting the vertices w_1 and w_2. Combining the definition of the triangulation with the case that $\langle w_1, w_2\rangle \in E$, we deduce that there exist exactly two vertices u^*, v^* such that $\langle w_1, w_2, u^*\rangle$ and $\langle w_1, w_2, v^*\rangle$ are two faces in G.

We claim that $\{u^*, v^*\} = \{u, v\}$. We assume that $\{u^*, v^*\} \neq \{u, v\}$. Without loss of generality, we suppose that $u^* \neq u, u^* \neq v$. Thus, by Observation 1.1, $P_{w_1} \cap P_{w_2} \cap P_{u^*} = \{p\}$. This implies that the point $\{p\}$ belongs to five

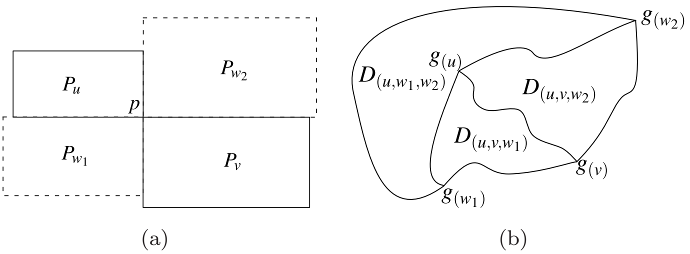

Figure 2

distinct rectangles $P_{w_1}, P_{w_2}, P_u, P_v, P_{u^*}$, which contradicts our assumption that $P = (P_v : v \in V)$ is a packing.

So there are four distinct faces in G which are $\langle u, v, w_1\rangle$, $\langle u, v, w_2\rangle$, $\langle w_1, w_2, u\rangle$ and $\langle w_1, w_2, v\rangle$ (see Figure 2 (b)). Note that the graph G is a disk triangulation graph. We can embed G onto the complex plane $\mathbb{C}$, denoted by g.

If $\langle x, y, z\rangle$ is a face in G, we denote by $D_{\langle x,y,z\rangle}$ the bounded component of $\mathbb{C} - \{g(\langle x, y\rangle) \cup g(\langle y, z\rangle) \cup g(\langle z, x\rangle)\}$, which is the interior of $g(\langle x, y, z\rangle)$.

Let

$$\gamma = g(\langle w_1, v\rangle) \cup g(\langle v, w_2\rangle) \cup g(\langle w_2, u\rangle) \cup g(\langle u, w_1\rangle).$$

It is clear that γ is a Jordan curve and

$$D_{\langle u,v,w_1\rangle} \cup D_{\langle u,v,w_2\rangle} \cup \{g(\langle u, v\rangle) - [g(u) \cup g(v)]\}$$

is the bounded component of $\mathbb{C}-\gamma$. So the open arc $g(\langle w_1, w_2\rangle)-[g(w_1)\cup g(w_2)]$ is contained in the unbounded component of $\mathbb{C}-\gamma$. Let W denote the unbounded component of $\mathbb{C} - \gamma$. Since the closed arc $g(\langle w_1, w_2\rangle)$ is a cross-cut in Jordan domain W, by the cross-cut theorem (Theorem 11-8, p. 119, [15]), we deduce that $W - g(\langle w_1, w_2\rangle) = W_1 \cup W_2$, where W_i, $i = 1, 2$, are two components of $W - g(\langle w_1, w_2\rangle)$; and

$$\partial W_1 = g(\langle w_1, u\rangle) \cup g(\langle u, w_2\rangle) \cup g(\langle w_1, w_2\rangle);$$

$$\partial W_2 = g(\langle w_1, v\rangle) \cup g(\langle v, w_2\rangle) \cup g(\langle w_1, w_2\rangle).$$

Thus one of two components of $W - g(\langle w_1, w_2\rangle)$ must be unbounded.

We assume without loss generality that W_2 is unbounded. This means that $D_{\langle w_1,w_2,v\rangle} \cap W_2 = \emptyset$ and $D_{\langle w_1,w_2,v\rangle} = W_1$. Hence

$$\mathbb{C} = W_1 \cup W_2 \cup D_{\langle u,v,w_1\rangle} \cup D_{\langle u,v,w_2\rangle} \cup \text{ some edges.}$$

For any edge $g(\langle x,y\rangle)\subset\mathbb{C}$, we have $D_{\langle w_1,w_2,v\rangle}\cap g(\langle x,y\rangle)=\emptyset$. Furthermore, we have

$$D_{\langle w_1,w_2,v\rangle}\cap W_i=\emptyset,\quad i=1,2,$$

and

$$D_{\langle w_1,w_2,v\rangle}\cap D_{\langle u,v,w_2\rangle}=\emptyset.$$

These imply $D_{\langle w_1,w_2,v\rangle}\cap\mathbb{C}=\emptyset$, which is a contradiction. Thus we complete the proof of Lemma 4.1. ∎

The following two lemmas for the smooth disks case appeared in [7]. For the sake of completeness we give their proofs here.

Lemma 4.2: *Let $P=\{P_v : v\in V\}$ be a packing of rectangles in the plane $\mathbb{C}$, and suppose that the contacts graph $G=(V;E)$ of P is a disk triangulation graph. Let $v_0\in V$ be some vertex, and let $N\subset V-\{v_0\}$ be the set of neighbors of v_0. Then there is a Jordan curve $\gamma\subset\bigcup_{v\in N}P_v-P_{v_0}$ which separates P_{v_0} from $\bigcup_{v\in V-(N\cup\{v_0\})}P_v$ in $\mathbb{C}$.*

Proof. From Lemma 4.1, it follows that, if $\langle u,v\rangle$ is an edge in G, the intersection of P_u and P_v is a segment. So we can construct an embedding of G in $\mathbb{C}$, denoted by f, such that the image of any edge $\langle v_1,v_2\rangle$ is contained in $P_{v_1}\cup P_{v_2}$ and is disjoint from all other rectangles in the packing (see Figure 3). For each $v\in V$ let C_v be the center point of the rectangle P_v, and for each edge $\langle v_1,v_2\rangle$ let p_{v_1,v_2} be the center point of the intersection segment of $P_{v_1}\cap P_{v_1}$. We may then write

$$f(\langle v_1,v_2\rangle)=\{(1-t)C_{v_1}+tp_{v_1,v_2} : 0\le t\le 1\}\cup\{(1-t)p_{v_1,v_2}+tC_{v_2} : 0\le t\le 1\}.$$

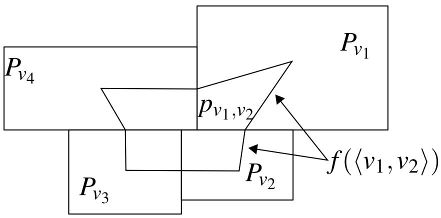

Figure 3

In the following we will view G as the 1-skeleton of a triangulation T of an open disk.

Let $\langle v_1, v_2, v_3\rangle$ be any triangle in T. For $j = 1, 2, 3$, let V_j be the neighbors of v_j in G. Clearly, for each $j = 1, 2, 3$, the vertices set $V_j' = V_j - \{v_1, v_2, v_3\}$ is connected. Since any pair of the sets V_1', V_2', V_3' intersect, the union $V' = V_1' \cup V_2' \cup V_3'$ is connected. Since G is connected and any path from a vertex $v \in V - \{v_1, v_2, v_3\}$ to a vertex in $\{v_1, v_2, v_3\}$ must intersect V', it follows that any two vertices in $G-\{v_1, v_2, v_3\}$ can be connected by a path in $G-\{v_1, v_2, v_3\}$.

If $(u_1, u_2, \ldots, u_n)$ is a path in $G-\{v_1, v_2, v_3\}$, then the path $\bigcup_{j=1}^{n-1} f(\langle u_j, u_{j+1}\rangle)$ is disjoint from $P_{v_1} \cup P_{v_2} \cup P_{v_3}$, and intersects both P_{u_1} and P_{u_n}. We conclude that every P_v, $v \in V - \{v_1, v_2, v_3\}$ is contained in the same connected component of $\mathbb{C} - (f(\langle v_1, v_2\rangle) \cup f(\langle v_2, v_3\rangle) \cup f(\langle v_3, v_1\rangle))$. Note that the set $f(\langle v_1, v_2\rangle)\cup f(\langle v_2, v_3\rangle)\cup f(\langle v_3, v_1\rangle)$ is a simple closed curve. Denote by B_{v_1,v_2,v_3} the component of $\mathbb{C} - (f(\langle v_1, v_2\rangle)\cup f(\langle v_2, v_3\rangle)\cup f(\langle v_3, v_1\rangle))$ that is disjoint from $\bigcup_{v\in V-\{v_1,v_2,v_3\}} P_v$. For any two distinct triangles $\langle v_1, v_2, v_3\rangle$, $\langle w_1, w_2, w_3\rangle$ in T, the intersection of the two Jordan curves $\partial B_{v_1,v_2,v_3}$, $\partial B_{w_1,w_2,w_3}$ is empty or consists of a single point or a segment. Therefore, either B_{v_1,v_2,v_3}, B_{w_1,w_2,w_3} are disjoint, or one is contained in the other. Suppose, without loss of generality, that $w_1 \notin \{v_1, v_2, v_3\}$. Then $\partial B_{w_1,w_2,w_3}$ intersects P_{w_1}, which is disjoint from the closure of B_{v_1,v_2,v_3}. We conclude that B_{w_1,w_2,w_3} is not contained in B_{v_1,v_2,v_3}. Similarly, B_{v_1,v_2,v_3} is not contained in B_{w_1,w_2,w_3}. Hence $B_{w_1,w_2,w_3} \cap B_{v_1,v_2,v_3} = \emptyset$.

Let $n_0, n_1, \ldots, n_{k-1}$ be the neighbors of v_0 in clockwise circular order around v_0, and let curve γ be the Jordan curve $\gamma = \bigcup_{j=0}^{k-1} f(\langle n_j, n_{j+1}\rangle)$, where $n_k \equiv n_0$. Then $\gamma \subset \bigcup_{v\in N} P_v$ and $\gamma \cap P_{v_0} = \emptyset$.

We say two distinct triangles $\langle v_1, v_2, v_3\rangle$, $\langle w_1, w_2, w_3\rangle$ in T are neighbors if they share an edge. Suppose that $\langle v_1, v_2, v_3\rangle$ is a triangle of T that does not contain v_0, and one of the neighboring triangles contains v_0, say $\langle v_0, n_j, n_{j+1}\rangle$. Then B_{v_1,v_2,v_3} and $B_{v_0,n_j,n_{j+1}}$ lie on opposite sides of the arc $f(\langle n_j, n_{j+1}\rangle)$. Consequently, B_{v_1,v_2,v_3} is not in the same connected component of $\mathbb{C} - \gamma$ as P_{v_0}. If $\langle v_1, v_2, v_3\rangle$ and $\langle w_1, w_2, w_3\rangle$ are two neighboring triangles that do not contain v_0, then it is clear that B_{v_1,v_2,v_3} and B_{w_1,w_2,w_3} are in the same connected component of $\mathbb{C} - \gamma$. Hence it easily follows that for every triangle

$\langle v_1, v_2, v_3\rangle$ that does not contain v_0, the set B_{v_1,v_2,v_3} is disjoint from the connected component of $\mathbb{C}-\gamma$ that contains P_{v_0}. This implies that γ separates P_{v_0} from $\bigcup_{v\in V-(N\cup\{v_0\})} P_v$, and the lemma follows since $\gamma \subset \bigcup_{v\in N} P_v - P_{v_0}$. ∎

LEMMA 4.3: *Let $P = \{P_v : v \in V\}$ and $G = (V; E)$ be as in Lemma 4.2, and let $u \in V$, $C \subset V - \{u\}$. Suppose that C is finite and u is contained in a finite component of $G - C$. Then $\bigcup_{v\in C} P_v$ separates P_u from the set of accumulation points of P.*

Proof. Let V_0 be the set of vertices that are contained in the same connected component of $G - C$ as u. Let $K \subset \mathbb{C} - \bigcup_{v\in C} P_v$ be a connected set that intersects P_u. For $w \in V$, let $N(w) \subset V - w$ denote the neighbors of w in G. From Lemma 4.2 it follows that for each $w \in V_0$ there is a Jordan curve $\gamma_w \subset \bigcup_{v\in N(w)} P_v - P_w$ that separates P_w from $\bigcup_{v\in V-(N(w)\cup\{w\})} P_v$.

Let Q_w denote the component of $\mathbb{C} - \gamma_w$ that contains P_w, and let $Q = \bigcup_{v\in V_0} Q_v$. Suppose that $p \in K \cap \partial Q_w$, where $w \in V_0$. Then $p \in K \cap \gamma_w$. Since K is disjoint from $\bigcup_{v\in C} P_v$ and $\gamma_w \subset \bigcup_{v\in N(w)} P_w$, we conclude that $p \in Q_{w'}$ with $w' \in V_0$. Thus $\partial Q_w \cap K \subset Q$ for every $w \in V_0$. Since V_0 is finite, we have $\partial Q \subset \bigcup_{v\in V_0} \partial Q_v$. The above implies that $\partial Q \cap K \subset Q$, and because Q is given, $\partial Q \cap K = \emptyset$. Hence $Q \cap K$ is a relatively open and relatively closed subset of K. As $Q \cap K \neq \emptyset$ and K is connected, we conclude that $K \subset Q$. Because each Q_v intersects finitely many of the sets in the packing P, the lemma follows. ∎

5. Duality

In this section we will show the connection between the α-Extremal Length and the α^*-Extremal Length. We will present some propositions of the locally finite, infinite, connected graph $G = (V, E)$. For any vertices $W \subseteq V$, ∂W denotes the set of vertices that are not in W but neighbor with some vertex in W. Suppose the metric $m_{id}(v) = 1$ for every vertex $v \in V$ and γ is a path in G. We denote by $CL(\gamma) \equiv \text{Length}_{m_{id}}(\gamma)$ the combinatorial length of the path γ. This means $CL(\gamma) = n$ if $\gamma = (v_1, v_2, \ldots, v_n)$.

PROPOSITION 5.1: *For any locally finite, infinite, connected graph $G = (V, E)$, let $\{v_0\}$ be a fixed vertex and $\{v_n : n = 1, 2, \ldots\}$ be infinite number of distinct vertices in $V \equiv V(G)$. For each $n \geq 1$, let γ_n be a path from v_0 to v_n. Then there exists a transient path $\gamma^* = (v_1^*, v_2^*, \ldots, v_n^*, \ldots)$ with the property: for* *each $j \geq 1$, there exists a path γ_{n_j} such that*

$$V(\gamma_j^*) \subseteq V(\gamma_{n_j}),$$

where $\gamma_j^* = (v_1^*, v_2^*, \ldots, v_j^*)$ is a finite sub-path of γ^* with $CL(\gamma_j^*) = j$.

Proof. Let $W_0 = \{v_0\}$. For every $k = 1, 2, \ldots,$ we define W_k inductively by setting $W_{k+1} = W_k \cup \partial W_k$. For a fixed integer $N \geq 1$, it is clear that $|W_N| < \infty$ since G is locally finite.

Let

$$S_N(v_0) = \{\gamma : \gamma \text{ is a path with initial point } v_0 \text{ and } CL(\gamma) = N\}.$$

Then $|S_N(v_0)| \leq |W_N|^{N-1}$. This implies that the total number of paths with the initial point v_0 and with the same combinatorial length is finite.

So we can rearrange the paths $\{\gamma_n\}$ according to their combinatorial length. Let $\{\gamma_n^1\}$ be the rearrangement of $\{\gamma_n\}$ according to their combinatorial length. This means that

$$CL(\gamma_1^1) \leq CL(\gamma_2^1) \leq \cdots \leq CL(\gamma_n^1) \leq \cdots.$$

Since $|S_N(v_0)|$ is finite, it is obvious that $\lim_{n\to\infty} CL(\gamma_n^1) = \infty$.

Now we will define a collection of subsequences $\{\gamma_n^k\}$ of $\{\gamma_n^1\}$ such that $\{\gamma_n^{k+1}\}$ is a subsequence of $\{\gamma_n^k\}$. Suppose that the paths $\{\gamma_n^1\}$ are written as follows:

$$\begin{aligned}
\gamma_1^1 =& (u_{11}^1,\ u_{12}^1,\ u_{13}^1, \ldots,\ u_{1p_1^1}^1);\\
\gamma_2^1 =& (u_{21}^1,\ u_{22}^1,\ u_{23}^1, \ldots,\ u_{2p_2^1}^1);\\
&\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots\\
\gamma_n^1 =& (u_{n1}^1,\ u_{n2}^1,\ u_{n3}^1, \ldots,\ u_{np_n^1}^1);\\
&\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}$$

where $p_n^1 = CL(\gamma_n^1)$ with $p_1^1 \leq p_2^1 \leq \cdots \leq p_n^1 \leq \cdots$ and $\lim_{n\to\infty} p_n^1 = \infty$.

Since $\{\gamma_n^1\}$ have the same initial point v_0 and G is locally finite, the total number of vertices which neighbor v_0 is finite. Thus we can extract an infinite subsequence $\{\gamma_n^2\}$ of $\{\gamma_n^1\}$ such that the second vertex in every path γ_n^2 is the same vertex. If the sequence γ_n^2 is written as follows:

$$\begin{aligned}
\gamma_1^2 =&(u_{11}^2,\ u_{12}^2,\ u_{13}^2, \ldots,\ u_{1p_1^1}^2);\\
\gamma_2^2 =&(u_{21}^2,\ u_{22}^2,\ u_{23}^2, \ldots,\ u_{2p_2^1}^2);\\
&\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots\\
\gamma_n^2 =&(u_{n1}^2,\ u_{n2}^2,\ u_{n3}^2, \ldots,\ u_{np_n^1}^2);\\
&\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}$$

then we have

$$v_0 = u_{11}^2 = u_{21}^2 = \cdots = u_{n1}^2 = \cdots;$$
$$u_{12}^2 = u_{22}^2 = \cdots = u_{n2}^2 = \cdots.$$

From the construction, it is clear that $CL(\gamma_n^2) \geq 2$ for each n and $\lim_{n\to\infty} CL(\gamma_n^2) = \infty$.

The general inductive step in the definition is now easy to formulate. We have the array

$$\begin{array}{cccc}
\gamma_1^1, & \gamma_2^1, & \ldots, & \gamma_n^1, \ \cdots\\
\gamma_1^2, & \gamma_2^2, & \ldots, & \gamma_n^2, \ \cdots\\
\vdots & \vdots & & \vdots\\
\gamma_1^k, & \gamma_2^k, & \ldots, & \gamma_n^k, \ \cdots\\
\vdots & \vdots & & \vdots
\end{array}$$

where each row is a subsequence of the row above. Furthermore, we have $CL(\gamma_n^k) \geq k$ for each n and $\lim_{n\to\infty} CL(\gamma_n^k) = \infty$.

Now consider the diagonal sequence $\{\gamma_1^1,\ \gamma_2^2,\ \ldots,\ \gamma_n^n,\ \ldots\}$ which is a subsequence of γ_n. Define the vertex

$$\begin{aligned}
v_n^* &= \text{the } n\text{-th vertex of the path } \gamma_n^n\\
&= u_{nn}^n.
\end{aligned}$$

Hence, from the above construction, it follows that the path

$$\gamma^* = (v_1^*,\ v_2^*,\ \ldots\ v_n^*,\ \ldots)$$

is the transient path we need. So the proposition is proved. ∎

THEOREM 5.1: *Let $\{v_0\}$ be a fixed vertex in the graph $G = (V, E)$. Write $\Gamma = \Gamma(v_0, \infty)$. Denote by Γ^* the collection of all subsets $C \subset V - \{v_0\}$ such that C intersects every $\gamma \in \Gamma$. If $EL^*_\alpha(\Gamma) = \infty$, then $EL_\alpha(\Gamma^*) = 0$.*

Proof. Because $\mathrm{EL}^*_\alpha(\Gamma) = \mathrm{EL}^*_\alpha(\{v_0\}, \infty) = \infty$, by Proposition 2.1 (taking $\beta(v) = \alpha(v)$ for each $v \in V$), there exists a metric $m_0 \in \mathcal{M}_\alpha(V)$ such that $d^*_{m_0}(\{v_0\}, \infty) = \infty$. Thus, for any positive number $L > 0$ and every $\gamma \in \Gamma$, we have

$$\text{Dual-Length}_{m_0}(\gamma) > L + 1.$$

For $v \in V$, define the **height** of v by

$$\begin{aligned} h(v) &= d^*_{m_0}(\{v_0\}, v) \\ &= \inf\{\text{Dual-Length}_{m_0}(\gamma) : \gamma \text{ is a path from } v_0 \text{ to } v\}. \end{aligned} \tag{5.1}$$

Let $I_v = [h(v) - \alpha(v)m_0(v),\, h(v)]$. For $t \in \mathbb{R}$, let V_t denote the set of vertices $v \in V$ such that

$$h(v) - \alpha(v)m_0(v) \le t \le h(v).$$

In order to prove Theorem 5.1, we need the following facts.

FACT 5.2: *Suppose $d^*_{m_0}(\{v_0\}, \infty) = \infty$. Then we have:*

1. *For each real number $M > 0$, there are only finitely many vertices $v \in V$ with $h(v) \le M$.*
2. *For each real number $M > 0$, there are only finitely many intervals I_v with $I_v \cap [0, M] \ne \emptyset$.*

Proof. If conclusion (1) is not true, there exist infinite distinct vertices $\{v_n : v_n \in V\}$ with $h(v_n) \le M$. By the definition of the weight function $h(v_n)$, we deduce that there exists the path γ_n from v_0 to v_n such that $\text{Dual-Length}_{m_0}(\gamma_n) \le M + 1$. By Proposition 5.1, we can find a transient path γ^* such that $\text{Dual-Length}_{m_0}(\gamma^*) \le M + 1$, which contradicts the assumption $d^*_{m_0}(\{v_0\}, \infty) = \infty$.

To obtain conclusion (2), we only need to show that there are finitely many vertices $v \in V$ satisfying $h(v) - \alpha(v)m_0(v) \le M$.

If (2) does not hold, then there exists an infinite sequence of vertices $\{v_n\}_{n=1}^\infty$ such that

$$h(v) - \alpha(v)m_0(v) \le M.$$

We claim that there exists a subsequence of vertices $\{u_p = v_{n_p}\} \subset \{v_n\}$ which satisfies that, for every pair vertices $u_i \neq u_j$ in such a subsequence,

$$\partial\{u_i\} \cap \partial\{u_j\} = \emptyset.$$

Recall that ∂W_0 denotes the set of vertices that are not in W_0 but neighbor with some vertex in W_0 for a finite subset W_0 of V.

Indeed, we define inductively a sequence $0 < n_1 < n_2 < \cdots$ of positive integers such that $\{u_p = v_{n_p}\}$ is such a sequence we need. For $k = 0, 1, 2, \ldots$, let W_k be defined inductively by $W_{k+1} = W_k \cup \partial W_k$. Let $n_1 = 1$ and $u_1 = v_{n_1}$. Suppose that $p > 1$, and that $n_1, n_2, \ldots, n_{p-1}$ have been defined. We set $W_0^{p-1} = \{v_{n_{p-1}}\}$. Since $G = (V, E)$ is locally finite, then

$$W_2^{p-1} = (W_0^{p-1} \cup \partial W_0^{p-1}) \cup \partial(W_0^{p-1} \cup \partial W_0^{p-1})$$

is a finite set (that is, $|W_2^{p-1}| < \infty$).

Thus, there exists an integer $n_p > n_{p-1}$ such that, for all $m \geq n_p$, $v_m \notin W_2^{p-1}$. Therefore, we choose $u_p = v_{n_p}$. The choice of $\{u_p\}$ shows that, for any $i \neq j$,

$$\partial W_0^i \cap \partial W_0^j = \emptyset,$$

where $W_0^i = \{u_i\}$ and $W_0^j = \{u_j\}$.

Therefore, we can find an infinite set of vertices $\{u_p\}$ with $h(u_p) - \alpha(u_p)m_0(u_p) \leq M$, and, for any $i \neq j$, we have $\partial W_0^i \cap \partial W_0^j = \emptyset$.

Since $h(u_p) - \alpha(u_p)m_0(u_p) < M + 1$, by using the definition of $h(\cdot)$, we conclude that there exists a path $\gamma_p = (v_0^p, v_1^p, \ldots, v_{l_p}^p)$, $v_0^p = v_0$ and $v_{l_p}^p = u_p$, such that

$$\text{Dual-Length}_{m_0}(\gamma_p) = \sum_{v \in \gamma_p} \alpha(v)m_0(v) - \alpha(u_p)m_0(u_p) < M + 1.$$

Let $\widetilde{\gamma}_p = (v_0^p, v_1^p, \ldots, v_{l_p-1}^p)$ be a sub-path of γ_p, and $w_p = v_{l_p-1}^p$. Then we have

$$h(\widetilde{\gamma}_p) \leq \text{Dual-Length}_{m_0}(\widetilde{\gamma}_p) < M + 1.$$

Note that $w_i \in \partial W_0^i$ and $\partial W_0^i \cap \partial W_0^j = \emptyset\,(i \neq j)$, where $W_0^i = \{u_i\}$ and $W_0^j = \{u_j\}$. We get $w_i \neq w_j$ for any $i \neq j$, which implies that we can find infinite distinct vertices $\{w_p\}$ with $h(w_p) < M + 1$. This contradicts conclusion (1). So conclusion (2) is proved. ∎

Fact 5.3: *For every $t \in [c_0, L]$, $V_t \in \Gamma^*$, where $c_0 = \alpha(v_0)m_0(v_0) + 1$ and Γ^* is the collection of all subsets $S \subset V - \{v_0\}$ such that S intersects every $\gamma \in \Gamma$.*

Proof. For every $t \in [c_0, L]$, by the definition of Γ^*, we only need to prove that $v_0 \notin V_t$ and $\gamma \cap V_t \neq \emptyset$ for every $\gamma \in \Gamma$.

Since $t > c_0 > h(v_0) = \alpha(v_0)m_0(v_0)$, from the definition of V_t, it follows that $v_0 \notin V_t$.

Suppose $\gamma_0 \cap V_t = \emptyset$ for some path $\gamma_0 \in \Gamma$. In the following, we will get a contradiction. Denote

$$V_1 = \{v \in \gamma_0 \ : \ h(v) < t\};$$

$$V_2 = \{v \in \gamma_0 \ : \ h(v) - \alpha(v)m_0(v) > t\}.$$

It is clear that $V_1 \cap V_2 = \emptyset$. Since $\gamma_0 \cap V_t = \emptyset$, we get $\gamma_0 = V_1 \cup V_2$.

Suppose that $\gamma_0 = (u_0, u_1, u_2, \ldots, u_n, \ldots)$ where $u_0 \equiv v_0$. Since

$$h(v_0) - \alpha(v_0)m_0(v_0) = 0,$$

we have $u_0 \notin V_2$, which implies $u_0 \in V_1$. Fact 5.2 implies that there are only finitely many vertices in γ_0 satisfying $h(v) \leq L$. Therefore, we can find a vertex $u_{n_0} \in \gamma_0$ such that $h(u_{n_0}) > L \geq t$. This shows that $u_{n_0} \notin V_1$. By using $\gamma_0 \cap V_t = \emptyset$ and $\gamma_0 = V_1 \cup V_2$, we have $u_{n_0} \in V_2$.

Let $\widetilde{\gamma}_0 = (u_0, u_1, \ldots, u_{n_0})$ be a sub-path of γ_0. Now we define an integer set B as follows:

$$B = \{1 \leq j \leq n_0 \ : \ u_j \in V_1 \cap \widetilde{\gamma}_0\}.$$

Now we let the integer k be the largest integer in the set B. Since $u_0 \in V_1$ and $u_{n_0} \in V_2$, we have $0 < k < n_0$. From the definition of k, it follows that $u_k \in V_1$, $u_{k+1} \in V_2$ and (u_k, u_{k+1}) is an edge in graph G. These imply

$$h(u_k) < t \tag{5.2}$$

and

$$t < h(u_{k+1}) - \alpha(v_{k+1})m_0(v_{k+1}). \tag{5.3}$$

By the definition of function $h(\cdot)$ and the fact that $(u_k, u_{k+1}) \in E(G)$, we get

$$h(u_{k+1}) \leq h(u_k) + \alpha(u_{k+1})m_0(u_{k+1}). \tag{5.4}$$

Combining (5.3) with (5.4), we obtain

$$h(u_k) \geq h(u_{k+1}) - \alpha(u_{k+1})m_0(u_{k+1}) > t,$$

which contradicts inequality (5.2). So Fact 5.3 is proved. ∎

Proof of Theorem 5.1, continued. Now let $m^* \in \mathcal{M}_\alpha(V)$ be any metric, and set

$$L^* = \inf\{\mathrm{Length}_{m^*}(S) \ : \ S \in \Gamma^*\}.$$

By Fact 5.3, we have $V_t \in \Gamma^*$ for $t \in [c_0, L]$. Define a function $f(t) = \sum_{v\in V_t} m^*(v)$ for $t \in [c_0, L]$. Since there are only finite intervals I_v satisfying $I_v \cap [0, L] \neq \emptyset$ ((2) of Fact 5.2), we know that the function $f(t)$ is a finite step function. That is, $f(t)$ is an integrable function on $[0, L]$. Thus

$$L^*(L - c_0) \leq \int_{c_0}^{L} \mathrm{Length}_{m^*}(V_t)\, dt \leq \int_0^L \mathrm{Length}_{m^*}(V_t)\, dt = \int_0^L \sum_{v\in V_t} m^*(v)\, dt.$$

For any $v \in V$, the set of t such that $v \in V_t$ is an interval of length $\alpha(v)m_0(v)$. Therefore, the above inequality yields

$$L^*(L - c_0) \leq \sum_{v\in V} m^*(v)\alpha(v)m_0(v) \leq \sqrt{\mathrm{area}_\alpha(m^*)}\sqrt{\mathrm{area}_\alpha(m_0)},$$

which gives

$$\frac{{L^*}^2}{\mathrm{area}_\alpha(m^*)} \leq \frac{\mathrm{area}_\alpha(m_0)}{(L - c_0)^2}.$$

Since $m_0 \in \mathcal{M}_\alpha(v)$ is a fixed metric, c_0 is a constant and L is an arbitrary positive number, we get

$$\mathrm{EL}_\alpha(\Gamma^*) = \sup\left\{ \frac{\left(\inf\{\mathrm{Length}_{m^*}(S) \ : \ S \in \Gamma^*\}\right)^2}{\mathrm{area}_\alpha(m^*)} : m^* \in \mathcal{M}_\alpha(V) \right\} = 0.$$

So Theorem 5.1 is proved. ∎

6. Parabolic rectangle packing

Proof of Theorem 1.3 (2), continued. By the assumption in Theorem 1.3, let $v_0 \in V$ be the vertex which satisfies $\mathrm{EL}^*_\alpha(\{v_0\}, \infty) = \infty$.

Now we distinguish two cases:

Case 1: The point ∞ is an accumulation point of the packing P.

Let $M(z)$ be a Möbius transformation such that $\{z \in \mathbb{C} : |z| \geq 1\}$ is contained in $\hat{P}_{v_0} = M(P_{v_0})$. Denote $\hat{P} = M(P) = \{\hat{P}_v = M(P_v) : v \in V\}$. Obviously, $M(\infty) \in D(0, 1)$.

Let Z be the set of accumulation points of the normalized packing $\hat{P}$. The assumption implies that the point $M(\infty) \in Z$. Our immediate goal is to verify that Z is connected in the extended complex plane $\hat{\mathbb{C}}$. Note that the graph G is locally finite. We can find a sequence of finite sunsets of V, denoted by

$V_1 \subset V_2 \subset \cdots$, such that $V = \bigcup_n V_n$. For each n, let Q_n denote the set of vertices in the infinite connected component of $G - V_n$, and let $\widetilde{Q}_n$ denote the closure of $\bigcup_{v \in Q_n} P_v$. Obviously we have $\widetilde{Q}_1 \supset \widetilde{Q}_1 \supset \cdots$, and each set $\widetilde{Q}_n$ is compact and connected in $\hat{\mathbb{C}}$. Since a nested intersection of compact connected sets is connected, it follows that Z is connected in $\hat{\mathbb{C}}$.

Let m be the v-metric on G defined by

$$m(v) = \begin{cases} \text{diameter}(\hat{P}_v) & \text{for } v \neq v_0, \\ 0 & \text{for } v = v_0. \end{cases}$$

From Lemma 3.2 and Fact 3.3, it follows that, for each $v \in V$, the set $\hat{P}_v$ is a $\alpha(v)/(200\pi(\alpha^2(v)+1))$-fat set. Since $\hat{P}_v \subset D(0,1)$ for $v \neq v_0$, we have

$$\begin{aligned} \text{area}_\alpha(m) &= \sum_{v \in V - \{v_0\}} \alpha(v) \left(\text{diameter}(\hat{P}_v)\right)^2 \\ &\leq 200 \sum_{v \in V - \{v_0\}} (\alpha^2(v)) + 1)\text{area}(\hat{P}_v) \\ &\leq C_1 \cdot \text{area}(D(0,1)) \\ &< \infty, \end{aligned}$$

where C_1 is a constant. So $m \in \mathcal{M}_\alpha(V)$.

Let C be any finite subset of $V - \{v_0\}$ such that v_0 is disjoint from the infinite connected component of $G - C$. The collection of all such subsets $C \subset V - \{v_0\}$ is denoted by Γ_1. From Lemma 4.3 it follows that the union $\bigcup_{v \in C} P_v$ separates P_{v_0} from Z. By the general Alexander's theorem (Theorem 16.1, p. 125, [15]), we deduce that there is a connected component of $\bigcup_{v \in C} P_v$ that separates P_{v_0} from Z. This implies that, for every $C \in \Gamma_1$,

$$\sum_{v \in C} m(v) \geq \text{diameter}\,(Z). \tag{6.1}$$

Recall that

$$\Gamma = \Gamma(\{v_0\}, \infty)$$

and

$$\Gamma^* = \{S \subset V - \{v_0\} \,:\, S \text{ intersects every } \gamma \in \Gamma\}.$$

Since $\text{EL}^*_\alpha(\Gamma) = \infty$, by Theorem 5.1 we get

$$\inf_{S \in \Gamma^*} \sum_{v \in S} m(v) = 0. \tag{6.2}$$

But every such $S \in \Gamma^*$ contains a finite $C \subset S$ such that v_0 is not in the infinite connected component of $G - C$ (for example, the neighbors of the connected component of $G - S$ containing v_0). This implies

$$\inf_{S \in \Gamma^*} \sum_{v \in S} m(v) \geq \inf_{C \in \Gamma_1} \sum_{v \in C} m(v).$$

Together with equation (6.2), we deduce that

$$\inf_{C \in \Gamma_1} \sum_{v \in C} m(v) = 0. \tag{6.3}$$

Therefore, (6.1) and (6.3) show that diameter $(Z) = 0$, which implies the set Z has only one point. Since $M(\infty) \in Z$, then $Z = \{M(\infty)\}$. Thus the packing $\hat{P} = M(P)$ is locally finite in $\hat{\mathbb{C}} - Z$. Equivalently, P is locally finite in $\mathbb{C}$.

Case 2: There exists a positive number R such that the packing P is contained in the disk $D(0, R)$.

Let Z_1 be the set of accumulation points of the packing P. Define $\eta\colon(0,\infty)\to\mathbb{C}$ by setting $\eta(t) = t\exp(\theta)$, where $\theta \in [0, 2\pi)$ is any fixed number. Then η is an open half-line. Since P is contained in the disk $D(0, R)$ and G is an open disk triangulation graph, we deduce that

$$\text{image}(\eta) \cap Z_1 \neq \emptyset.$$

Thus $|Z_1| = \infty$. But, by the same argument of Case 1, we get $|Z_1| = 1$, which is a contradiction.

Hence we complete the proof of Theorem 1.3. ∎

7. Appendix

For the sake of completeness, in this section we will construct an α-EL parabolic rectangle packing that is not locally finite in the complex plane $\mathbb{C}$.

Example 7.1: Define a constant u $(= \pi^2/6)$ with

$$u = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Step 1. Let

$$R_1 = [-u, u] \times [-1, 1]$$

be the center rectangle in the complex plane $\mathbb{C}$ (see Figure 4).

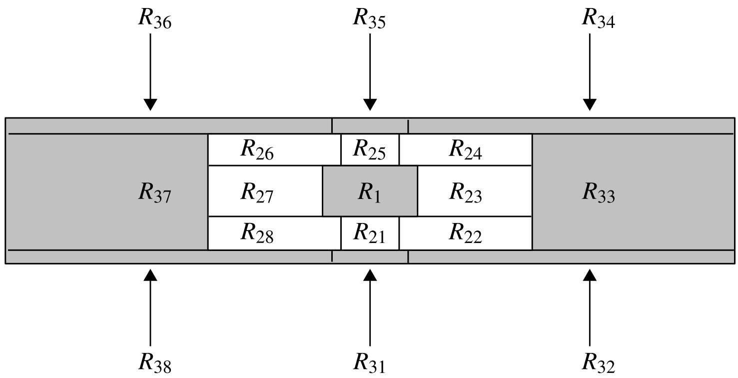

Figure 4

STEP 2. For each $n \geq 2$, we construct 8 rectangles as follows (see Figure 5). For $i = 1, 2, \ldots, 8$, let

$$R_{ni} = [a_{ni}, b_{ni}] \times [c_{ni}, d_{ni}]$$

be rectangles in $\mathbb{C}$, where

$$a_{n1} = -\sum_{k=1}^{n} \frac{1}{k^2}; \qquad b_{n1} = -a_{n1};$$
$$c_{n1} = -\sum_{k=1}^{n} \frac{1}{k^2}; \qquad d_{n1} = c_{n1} + \frac{1}{n^2} = -\sum_{k=1}^{n-1} \frac{1}{k^2},$$

and

$$a_{n3} = u + u\sum_{k=1}^{n-1} k^2; \qquad b_{n3} = a_{n3} + un^2 = u + u\sum_{k=1}^{n} k^2;$$
$$c_{n3} = -\sum_{k=1}^{n} \frac{1}{k^2}; \qquad d_{n3} = -c_{n3},$$

and

$$a_{n2} = b_{n1}; \quad b_{n2} = b_{n3};$$
$$c_{n2} = c_{n1}; \quad d_{n2} = -d_{n1},$$

and

$$\begin{aligned}
a_{n4} &= a_{n2}; & b_{n4} &= b_{n2}; \\
c_{n4} &= -d_{n2}; & d_{n4} &= -c_{n2}. \\
a_{n5} &= a_{n1}; & b_{n5} &= b_{n1}; \\
c_{n5} &= c_{n4}; & d_{n5} &= d_{n4}. \\
a_{n6} &= -b_{n4}; & b_{n6} &= -a_{n4}; \\
c_{n6} &= c_{n4}; & d_{n6} &= d_{n4}. \\
a_{n7} &= a_{n6}; & b_{n7} &= -a_{n3}; \\
c_{n7} &= c_{n3}; & d_{n7} &= d_{n3}. \\
a_{n8} &= a_{n6}; & b_{n8} &= a_{n1}; \\
c_{n8} &= c_{n1}; & d_{n8} &= d_{n1}.
\end{aligned}$$

(See Figure 5).

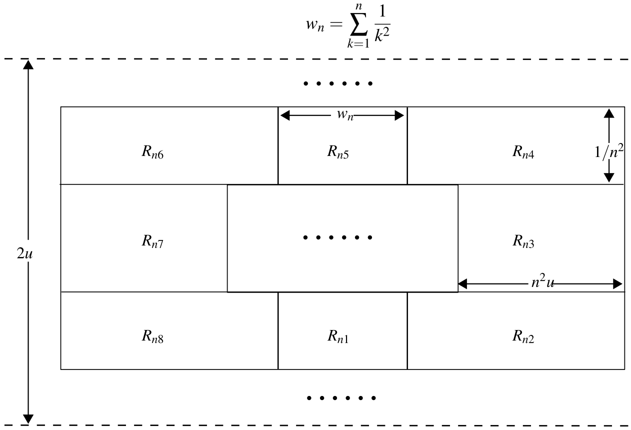

Figure 5

STEP 3. Denote

$$P = R_1 \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{8} R_{ni} \right).$$

Then P is an infinite rectangle packing the complex plane $\mathbb{C}$. It is clear that the contacts graph of P is a triangulation of an open topological disk (see Figure 5 and Figure 6).

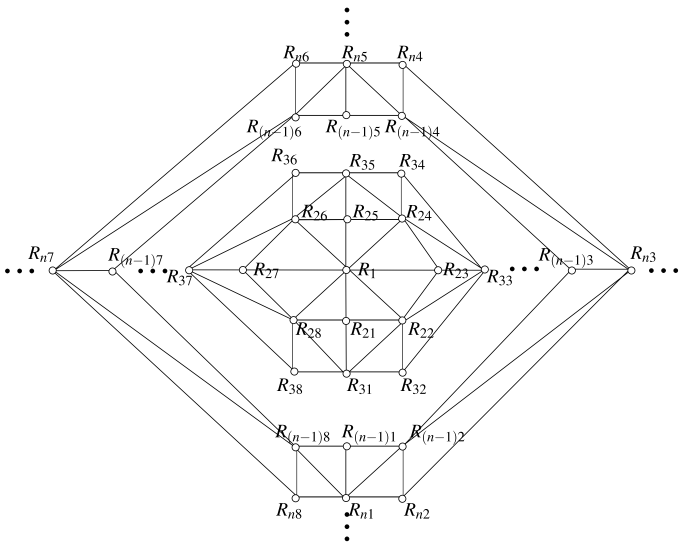

Figure 6

It is obvious that the points $(0,u)$ and $(0,-u)$ are accumulation points of the packing P. Therefore, the packing P is not locally finite in the plane.

Now, we show that the packing P is an α-EL parabolic packing.

Let $\alpha : P \to (0,1]$ be an assignment of weights to the vertices of packing P with $\alpha(R) = H(R)/W(R)$, where $H(R)$ is the height of the rectangle R and $W(R)$ its width and $H(R) \leq W(R)$. It clear that

$$\alpha(R_1) = 1/u \leq 1.$$

For $n \geq 2$, from the construction of the packing P, it follows that, for $1 \leq i \leq 8$,

$$\alpha(R_{ni}) \leq \frac{2}{n^2} \leq 1. \tag{7.1}$$

Define the metric m_{id} with

$$m_{id}(R) = 1 \quad \text{for each } R \in P.$$

Now we will show that $m_{id} \in \mathcal{M}_\alpha(P)$. Using the construction of packing P and (7.1), we obtain

$$\begin{aligned}\mathrm{area}_\alpha(m_{id}) =& \alpha(R_1) + \sum_{n=2}^{\infty}\sum_{i=1}^{8} \alpha(R_{ni}) m_{id}^2(R_{ni}) \\ \leq& \frac{1}{u} + 16\sum_{n=2}^{\infty}\frac{1}{n^2} \\ <& \infty.\end{aligned}$$

Therefore $m_{id} \in \mathcal{M}_\alpha(P)$. Since $m_{id}(R) = 1$ for each rectangle $R \in P$, we have

$$d_{m_{id}}(\{R_1\}, \infty) = \infty.$$

Proposition 2.1 (taking $\beta(R) = 1$ for each $R \in P$) implies that the packing P is an α-EL parabolic packing.

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