SCIENCE CHINA
Mathematics

• ARTICLES •

February 2010 Vol. 53 No. 2: 275–288
doi: 10.1007/s11425-009-0125-4

Schwarz's lemma for the circle packings with the general combinatorics

HUANG XiaoJun[1,*], LIU JinSong[2] & SHEN Liang[2]

[1] *College of Mathematics and Physics, Chongqing University, Chongqing* 400044, *China;*
[2] *Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academic of Sciences, Beijing* 100190, *China*
Email: hxj@cqu.edu.cn, liujsong@math.ac.cn, shenl@amss.ac.cn

Received November 6, 2007; accepted March 31, 2009; published online September 7, 2009

Abstract Rodin (1987) proved the Schwarz's lemma analog for the circle packings based on the hexagonal combinatorics. In this paper, we prove the Schwarz's lemma for the circle packings with the general combinatorics and our proof is more simpler than Rodin's proof. At the same time, we obtain a rigidity property for those packings with the general combinatorics.

Keywords circle packing, discrete extremal length, the maximum principle, hyperbolic geometry

MSC(2000): 52C15, 30C35, 30C62, 30G25

Citation: Huang X J, Liu J S, Shen L. Schwarz's lemma for the circle packings with the general combinatorics. Sci China Math, 2010, 53(2): 275–288, doi: 10.1007/s11425-009-0125-4

1 Introduction

In the following, a *disk packing* is a collection of closed geometric disks, with disjoint interior, in the plane $\mathbb{C}$, or the Riemann sphere $\hat{\mathbb{C}}$. An *interstice* of a disk packing P is a bounded connected component of the complement P, and an interstice whose closure intersects only three disks in the packing is a *triangular interstice*. The *support*, $\mathrm{supp}(P)$, of P is defined as the union of the disks in P and all bounded interstices of P.

Let P and $\tilde{P}$ be two finite disk packings in $\mathbb{C}$. If there is an orientation preserving homoeomorphism $h : \mathrm{supp}(P) \to \mathrm{supp}(\tilde{P})$ such that $h(P) = \tilde{P}$, then we shall say that P and $\tilde{P}$ are *isomorphism* between them. It is clear that an isomorphism h induces a bijection between the disks of P and the disks of $\tilde{P}$. An isomorphism of packings is essentially a combinatorial notion. One can determine if two packings are isomorphic, by looking at the pattern of tangency of the disks, etc.

It will be notationally convenient to work with indexed packings: an indexed disk packing $P = \{P(v) : v \in V\}$ is just a packing in which the disks are indexed by some set V. The *nerve*, or *tangency graph* $G = G(P)$, of an indexed packing $P = (P(v) : v \in V)$ is the graph on the vertex set V in which $[u, v]$ is an edge if and only if $P(v)$ and $P(u)$ intersect (tangentially). A graph is called a *closed disk triangulation graph* if it is equal to the one-skeleton of a triangulation T of the closed topological disk.

Of fundamental importance is the Circle Packing Theorem [7], which asserts that for any finite planar graph $G = (V, E)$ there is a disk packing $P = (P(v) : v \in V)$ in $\mathbb{C}$ whose tangency graph is G. Moreover,

*Corresponding author

when G is the one-skeleton of a triangulation of S^2, the packing is unique up to Möbius transformations. Some later proofs of the circle packing theorem include [1, 3, 8, 14].

Now suppose that $P = (P(v) : v \in V)$ is a finite packing in $\mathbb{C}$. The disks of P that intersect the boundary of the support of P is called *boundary disks* and ∂V denote the boundary vertices, that is, $v \in \partial V$ if and only if $P(v)$ is a boundary disk. It is a consequence of the circle packing theorem that there is an isomorphic disk packing $\tilde{P}$, which is contained in the closed unit disk $\overline{U}$, such that the boundary disks of $\tilde{P}$ are all internally tangent to the unit circle ∂U.

Let D be some bounded simply connected domain in $\mathbb{C}$, and p_0 be some interior point of D. For each n, let P^n be a disk packing in D, in which all bounded interstices are triangular (or $G(P^n)$ is a *closed disk triangulation graph*). Assume that there is a sequence of positive numbers δ_n with limit 0 such that: (i) the radius of any disk in the packing P^n is at most δ_n, and (ii) any boundary disk in the packing P^n is within δ_n from the boundary of D. Denote by P_0^n a selected disk of the packing P^n which (contains or) is the closest to p_0.

As mentioned above, the circle packing theorem implies that there is an isomorphic packing $\tilde{P}^n$ in the closed unit disk, with boundary disks all tangent to the unit circle ∂U. Let $f_n : \mathrm{supp}(P^n) \to \mathrm{supp}(\tilde{P}^n)$ be an isomorphism of P^n and $\tilde{P}^n$. Normalize $\tilde{P}^n$ by Möbius transformations preserving U so that the disk $\tilde{P}_0^n$ of $\tilde{P}^n$ corresponding to P_0^n is centered at 0.

In his talk [15], Thurston conjectured that the sequence of functions f_n converges to the Riemann mapping from D to U, if the packings P^n are taken as sub-packing of scaled copies of the infinite hexagonal circle packing. The conjecture was later proved by Rodin and Sullivan [11]. Later, the Schwarz lemma analog was proved in [10, Theorem 6.2 (p. 286)] by Rodin: Given a compact set $K \subset D$, there is an absolute constant M_K such that when n is sufficiently large, $\mathrm{radius}(\tilde{P}^n(v))/\mathrm{radius}(P^n(v)) \leqslant M_K$ for all vertices v with $P^n(v) \cap K \neq \emptyset$.

In [6], a new proof was given to the Rodin-Sullivan theorem by He Zhengxu and Oded Schramm, which avoided restrictions on the combinatorics of P^n. There is a natural problem: Whether the Schwarz lemma analog is still true for the He-Schramm theorem? In this paper, we confirm this problem and obtain the following

Theorem 1.1. *Let $D, \tilde{D} \subset \mathbb{C}$ be two bounded simply connected domains, and let p_0 be some point in D. For each n, let P^n be a disk packing in D, with $G(P^n)$ being a closed disk triangulation graph, let $\tilde{P}^n$ be an isomorphic packing in $\tilde{D}$.*

Let δ_n be a sequence of positive numbers, tending to zero, and assume that the diameters of the disks in every P^n are less than δ_n, and for each boundary disk $P^n(v)$ of P^n the distance from it to ∂D is less than δ_n. Suppose that for each n the point p_0 is contained in the support of P^n. Let K be a compact subset of D. Then for sufficient large $n \in \mathbb{N}$,

$$\mathrm{radius}(\tilde{P}^n(v)) \leqslant M_K \mathrm{radius}(P^n(v)) \tag{1.1}$$

for all vertices v of $G(P^n)$ with $P^n(v) \cap K \neq \emptyset$, where M_K is a constant only depending on the compact subset K.

In [10] (or [11]), when P^n are the subpackings of hexagonal packing with all circles having the same radius, the radius of the circles in corresponding packing $\tilde{P}^n$ have the rigidity property. In this paper, we also obtain a rigidity property analog as follows:

Theorem 1.2. *Let the situation be as in Theorem 1.1. Let K be a compact subset of D and v_0^n, v_1^n be two vertices of $G(P^n)$ with $P^n(v_0^n) \cap K \neq \emptyset$ and v_1^n neighboring v_0^n. Then for sufficient large $n \in \mathbb{N}$,*

$$\frac{1}{C}\frac{\mathrm{radius}(P^n(v_1^n))}{\mathrm{radius}(P^n(v_0^n))} \leqslant \frac{\mathrm{radius}(\tilde{P}^n(v_1^n))}{\mathrm{radius}(\tilde{P}^n(v_0^n))} \leqslant C\frac{\mathrm{radius}(P^n(v_1^n))}{\mathrm{radius}(P^n(v_0^n))}, \tag{1.2}$$

where C is a constant only depending on K.

Notational conventions. Throughout the paper, G will denote a graph, $E = E(G)$ the set of edges in G, and $V = V(G)$ the set of vertices. If P is a disk packing which realizes G, then for any subset of vertices $W \subseteq V$, we denote $P(W) = \bigcup_{v \in W} P(v)$. For a circle c, we denote by $V_c(P)$ the set of vertices v

for which $P(v) \cap c \neq \emptyset$. For any $r > 0$, we denote by $c(r)$ the circle of radius r centered at 0, and $D(r)$ the closed disk bounded by $c(r)$. For a disk D, we will denote $\rho(D)$ to be its euclidean radius. Finally, we denote by U the unit disk (i.e., $U = \{z \in \mathbb{C} \mid |z| < 1\}$).

2 Topological behavior

In this section, we present a topological property which will be used in our proof. The proof, included here for completeness, is elementary and well known.

Lemma 2.1. *Let D be a domain in the plane $\mathbb{C}$ with the point $z_0 \in D$. Let $D_n \subset D$ be a sequence of domains contained in D, $z_0 \in D_n$ and for any point $z \in \partial D_n$, $\operatorname{dist}(z, \partial D) < \delta_n$ with $\delta_n \to 0$ as $n \to \infty$. Then for every compact subset $K \subset D$, there is $K \subset D_n$ for sufficiently large n.*

Proof. Otherwise, there is a compact subset K_0 of D, a subsequence of domains D_{n_k} such that $K_0 \subsetneq D_{n_k}$, that is, there exists a sequence of points $\{z_{n_k}\}$ with $z_{n_k} \in K_0$ and $z_{n_k} \notin D_{n_k}$. Without loss of the generality, we may assume that $z_{n_k} \to z_1 \in K_0$. We set $C_r = \{z \,|\, \operatorname{dist}(z, z_1) \leqslant r < |z_0 - z_1|\} \subset D$ and choice a fixed Jordan curve $J \subset D$ with joining between z_0 and z_1. It is clear that $z_{n_k} \in C_r$ for sufficient large k. Setting $d = \operatorname{dist}(J \cup C_r, \partial D) > 0$. Because $z_0 \in D_{n_k}$ and $z_{n_k} \notin D_{n_k}$, we can easily obtain the points $z'_{n_k} \in (\overline{z_{n_k} z_1} \cup J) \cap \partial D_{n_k}$ where $\overline{z_{n_k} z_1}$ denotes the segment joining between z_{n_k} and z_1. Hence we get $\operatorname{dist}(z'_{n_k}, \partial D) \geqslant d$ where $z'_{n_k} \in \partial D_{n_k}$ which contradicts that for any point $z \in \partial D_n$, $\operatorname{dist}(z, \partial D) \to 0$ as $n \to \infty$. This completes the proof of Lemma 2.1.

Remark. From the above lemma, it is easy to see that $D = \bigcup_{n=1}^{\infty} D_n$ and it is a special of Carathéodory domain convergence.

3 The maximum principle

Various forms of the maximum principle, some weaker, some stronger, have been suggested and used by many authors. Here we will present it in a way suitable to our applications and the general disk patterns case due to He [4].

Let a graph $G = (V, E)$ be given. A vertex v_0 in G is called *interior* if there is a closed chain of neighboring vertices $v_1, v_2, \ldots, v_l$. Otherwise, it is called a *boundary* vertex. If G is the one-skeleton of a triangulation of a closed topological disk, then a vertex is the interior vertex precisely when it lies in the interior of the closed topological disk.

Lemma 3.1 (Maximum principle). *Let G be a finite graph. Let P and $\tilde{P}$ be disk packings in $\mathbb{C}$ which realize G. Then the maximum (or minimum) of $\rho(\tilde{P}(v))/\rho(P(v))$ is attained at a boundary vertex.*

Proof. Let v_0 be an interior vertex. Let $v_1, v_2, \ldots, v_l, v_{l+1} = v_1$ be the chain of neighboring vertices. Let Q be a disk packing which realizes the graph G. Denote by A_j be the center of the disk $Q(v_j)$, $1 \leqslant j \leqslant l+1$. Let β_k be the angle $\angle A_k A_0 A_{k+1}$ (see Figure 1). Since v_0 is interior, we have $\sum_{k=1}^{l} \beta_k = 2\pi$. On the other hand, $\beta_k = \arccos(g(\rho(Q(v_0)); \rho(Q(v_k)), \rho(Q(v_{k+1}))))$, where

$$g(x; y, z) = \frac{(x+y)^2 + (x+z)^2 - (y+z)^2}{2(x+y)(x+z)}.$$

By a simple computation, it is easy to see that β_k is a nondecreasing function of $\rho(Q(v_k))$ (and of $\rho(Q(v_{k+1}))$). Hence $\sum_{k=1}^{l} \beta_k$ is strictly increasing in $\rho(Q(v_k))$. The maximum principle follows immediately. This completes the proof of Lemma 3.1.

The version of the maximum principle also holds for disk packings in the hyperbolic disk U. For a closed disk D in U, denote by $\rho_{\text{hyp}}(D)$ its hyperbolic radius. Let D_0 be a fixed closed disk in U centered at 0, and let D_1 be a variable closed disk which tangents to D_0. For any real number γ and any disk D, we define $\gamma D = \{\gamma z \mid z \in D\}$. With these notations, we have

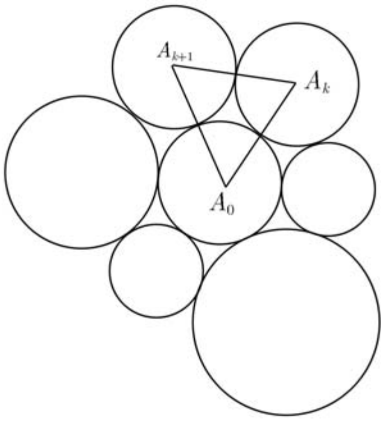

Figure 1 The configuration of the disks $Q(v_j)$

Proposition 3.2. *$\rho_{\rm hyp}(D_1)$ is a strictly increasing function of $\rho(D_1)$, and for any real number $0<\gamma<1$, we have*

$$\frac{\rho_{\rm hyp}(\gamma D_1)}{\rho_{\rm hyp}(\gamma D_0)}<\frac{\rho_{\rm hyp}(D_1)}{\rho_{\rm hyp}(D_0)}.$$

Proof. The first part of Proposition 3.2 is clear, so we prove only the second part of it. Without loss of generality, we assume that the disk D_1 center at the positive real axis. Let $p(u)=\ln\frac{1+u}{1-u}-\frac{2u}{1+u^2}$ for $u\in[0,1)$. Differentiating $p(u)$ about u, we get $p'(u)=[\frac{2}{1-u^2}-\frac{2}{(1+u^2)^2}]+\frac{2u^2}{(1+u^2)^2}>0$ for $0\leqslant u<1$. Because $p(0)=0$, then $p(u)>0$ when $0<u<1$. Set $h(u)=\frac{1-u^2}{u}\ln\frac{1+u}{1-u}$ with $0<u<1$. By a simple computation, we can obtain $h'(u)=\frac{2}{u}-\frac{u^2+1}{u^2}\ln\frac{1+u}{1-u}$. Noting $p(u)>0$, it follows that $h'(u)<0$ $(0<u<1)$. That is to say that $h(u)$ is strictly decreasing in $0<u<1$. Let

$$f(t)=\frac{\int_0^t\frac{2dx}{1-x^2}}{\int_0^{\gamma t}\frac{2dx}{1-x^2}},\quad \text{where } 0<\gamma<1 \text{ and } 0<t<1.$$

Differentiating above equation about t, we get

$$\frac{t}{2}\left(\int_0^{\gamma t}\frac{2dx}{1-x^2}\right)^2f'(t)=\frac{t}{1-t^2}\ln\frac{1+\gamma t}{1-\gamma t}-\frac{\gamma t}{1-(\gamma t)^2}\ln\frac{1+t}{1-t}.$$

Combining the property of the function $h(u)$ and above equation, we can obtain easily $f'(t)>0$ when $0<t<1$. Let ρ_i denote the euclidean radius of D_i for $i=0,1$. Since the disks D_0 and D_1 are contained in U and D_1 tangent to D_0, it follows that $\rho_0+2\rho_1<1$. According that the function $f(t)$ strictly increases in $0<t<1$, we have

$$\frac{\int_0^{\rho_0+2\rho_1}\frac{2dx}{1-x^2}}{\int_0^{\gamma(\rho_0+2\rho_1)}\frac{2dx}{1-x^2}}>\frac{\int_0^{\rho_0}\frac{2dx}{1-x^2}}{\int_0^{\gamma\rho_0}\frac{2dx}{1-x^2}}\Rightarrow\frac{\int_{\rho_0}^{\rho_0+2\rho_1}\frac{2dx}{1-x^2}}{\int_0^{\rho_0}\frac{2dx}{1-x^2}}>\frac{\int_{\gamma\rho_0}^{\gamma\rho_0+2\gamma\rho_1}\frac{2dx}{1-x^2}}{\int_0^{\gamma\rho_0}\frac{2dx}{1-x^2}},$$

that is,

$$\frac{2\rho_{\rm hyp}(D_1)}{\rho_{\rm hyp}(D_0)}>\frac{2\rho_{\rm hyp}(\gamma D_1)}{\rho_{\rm hyp}(\gamma D_0)}.$$

So Proposition 3.2 is proved.

Combining Proposition 3.2 with Lemma 3.1, we deduce

Lemma 3.3 (Maximum principle in the hyperbolic plane). *Let G, P and $\tilde{P}$ be as in Lemma 3.1. Assume that the disks of P and $\tilde{P}$ are contained in U. Then*

(a) *The maximum of $\rho_{\rm hyp}(\tilde{P}(v))/\rho_{\rm hyp}(P(v))$, if >1, is never attained at an interior vertex;*

(b) *In particular, if the inequality $\rho_{\rm hyp}(\tilde{P}_v)\leqslant\rho_{\rm hyp}(P_v)$ is true for each boundary vertex, then it holds for all vertices of G.*

Proof. The part (b) immediately follows from (a), so we only proof the part (a). If the part (a) fails, then there exists an interior vertex $v_0 \in V(G)$ such that

$$\frac{\rho_{\rm hyp}(\tilde{P}(v_0))}{\rho_{\rm hyp}(P(v_0))} = \max_{v\in V(G)} \frac{\rho_{\rm hyp}(\tilde{P}(v))}{\rho_{\rm hyp}(P(v))} > 1.$$

Hence, by a rigid motion of preserving the unit disk U (it also preserves the hyperbolic metric), we may assume that the disks $P(v_0)$ and $\tilde{P}(v_0)$ center at 0. It is clear that the euclidean radius of $P(v_0)$ is less than the euclidean radius of $\tilde{P}(v_0)$. Let $\gamma = \rho(P(v_0))/\rho(\tilde{P}(v_0)) < 1$ $(0 < \gamma < 1)$ and consider the disk packings P and $\gamma\tilde{P}$. Suppose v_1 be any vertex neighboring v_0. According to the supposition of v_0 and Proposition 3.2, we have

$$\frac{\rho_{\rm hyp}((\gamma\tilde{P})(v_0))}{\rho_{\rm hyp}((\gamma\tilde{P})(v_1))} > \frac{\rho_{\rm hyp}(\tilde{P}(v_0))}{\rho_{\rm hyp}(\tilde{P}(v_1))} \geqslant \frac{\rho_{\rm hyp}(P(v_0))}{\rho_{\rm hyp}(P(v_1))}.$$

Combining $\rho((\gamma\tilde{P})(v_0)) = \rho(P(v_0))$, the first part of Proposition 3.2 and the proof of Lemma 3.1, the contradiction follows immediately.

To extend the lemma, we make the following definition of $\rho_{\rm hyp}(D)$ for a closed disk D in $\mathbb{C}$ which intersects U. The *dihedral angle* of a pair of intersection disks D_1 and D_2 is defined to be the angle in $[0,\pi]$ between the clockwise tangent of ∂D_1 and the counterclockwise tangent of ∂D_2 at a point of $\partial D_1 \cap \partial D_2$ (see Figure 2). If D in contained in U, we define $\rho_{\rm hyp}(D)$ to be the hyperbolic radius as before. If D intersects $\hat{\mathbb{C}} - U$, let $\alpha = \pi - \beta(D) \in [0,\pi]$ where $\beta(D)$ denote the dihedral angle between D and U. We define $\rho_{\rm hyp}(D)$ to be the symbol ∞^α. In particular, $\rho_{\rm hyp}(D) = \infty^0$ if D is internally tangent to the unit circle ∂U. We make the convention that for any angles $\alpha_2 \geqslant \alpha_1 \geqslant 0$ and for any real number a, we have: $\infty^{\alpha_2} \geqslant \infty^{\alpha_1} > a$ and $\infty^{\alpha_1}/a = 0$. With the same proof, we have the following lemma.

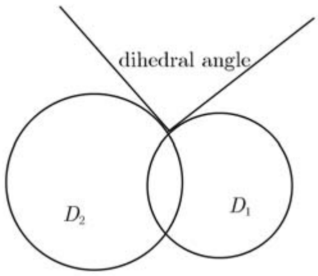

Figure 2 The dihedral angle

Lemma 3.4. *Let G be a finite graph. The disk packings P and $\tilde{P}$ realize G, all disks of P have non-empty intersection with U and all disks of $\tilde{P}$ are contained in U. If the inequality $\rho_{\rm hyp}(\tilde{P}(v)) \leqslant \rho_{\rm hyp}(P(v))$ is true for each boundary vertex v, then it holds for all vertices of G.*

Proof of Theorem 1.1. Let K be a compact subset in D, v be a vertex of $G(P^n)$ with $P^n(v) \cap K \neq \emptyset$. Suppose that $d = \mathrm{dist}(K, \partial D) > 0$ and σ is a disk with radius $d/2$ and centering at the centre of $P^n(v)$. It is clear that $\sigma \subseteq K^*$ where $K^* = \{z \in D \mid \mathrm{dist}(z,K) \leqslant (3d)/4\}$ for sufficient large n (because $\rho(P^n(v)) < \delta_n$). So, by Lemma 2.1, it follows that $\sigma \subset \mathrm{supp}(P^n)$ for sufficient large number n. Let $M = \mathrm{diameter}(\tilde{D})$. It is clear that all disks of $\tilde{P}^n$ are contained in the disk $\tilde{\sigma}$ which center at the centre of the disk $\tilde{P}^n(v)$ with radius $2M$.

Consider the subgraph $G_1^n \subset G^n$ consisting of the vertices v^* for which $P^n(v^*) \cap \sigma \neq \emptyset$ and the corresponding edges. Since $\sigma \subset \mathrm{supp}(P^n)$, it is easy to see that $P^n(v^*) \cap \partial\sigma \neq \emptyset$ if v^* is a boundary vertex of G_1^n. So, all disks $\tilde{P}^n(v^*)$, $v^* \in V(G_1^n)$, are contained in the interior of the disk $\tilde{\sigma}$. Hence the inequality $\rho_{\rm hyp}((1/\rho(\tilde{\sigma}))\tilde{P}^n(v^*)) \leqslant \rho_{\rm hyp}((1/\rho(\sigma))P^n(v^*))$ is held for each boundary vertex v^* in $V(G_1^n)$. Using Lemma 3.4, we deduce that for each vertex $v^* \in V(G_1^n)$,

$$\rho_{\rm hyp}\bigg(\frac{1}{\rho(\tilde{\sigma})}\tilde{P}^n(v^*)\bigg) \leqslant \rho_{\rm hyp}\bigg(\frac{1}{\rho(\sigma)}P^n(v^*)\bigg).$$

In particular, for the vertex v,

$$\rho_{\text{hyp}}\left(\frac{1}{\rho(\tilde{\sigma})}\tilde{P}^n(v)\right) \leqslant \rho_{\text{hyp}}\left(\frac{1}{\rho(\sigma)}P^n(v)\right). \tag{3.1}$$

Since the disks $(1/\rho(\tilde{\sigma}))\tilde{P}^n(v), (1/\rho(\sigma))P^n(v)$ are all contained in $(1/2)U$ for sufficient large n (because $\rho(P^n(v)) < \delta_n$), it follows that their euclidean radii are comparable with their hyperbolic radii. So by (3.1), there is a universal constant $C_0 > 0$ such that,

$$\frac{1}{\rho(\tilde{\sigma})}\rho(\tilde{P}^n(v)) \leqslant C_0 \frac{1}{\rho(\sigma)}\rho(P^n(v)).$$

Noting that $\rho(\tilde{\sigma}) = 2M$ and $\rho(\sigma) = d/2$, then

$$\rho(\tilde{P}^n(v)) \leqslant \frac{C_0(4M)}{d}\rho(P^n(v)).$$

So Theorem 1.1 is proved.

4 Discrete extremal length

We will make extensive use of the discrete extremal length in graphs. The notion resembles the extremal length of curve families in Riemann surfaces, and was introduced and studied by Cannon and others (see [2, 5, 12]). (A further extension to the idea of "transboundary extremal length" appeared in [13]) Below we will recall the basics. For details, we refer to [5].

Let $G = (V, E)$ be a graph and let V be the set of vertices. A *vertex metric* in the graph is just a function $\eta : V \to [0, +\infty)$. The *area* of the metric η is defined to be

$$\text{area}(\eta) = \sum_{v\in V} \eta(v)^2. \tag{4.1}$$

The collection of all metrics η on G with $0 < \text{area}(\eta) < \infty$ will be denoted $\mathscr{M}(G)$. Any subset of vertices will be called a *vertex curve*. The *η-length* of a vertex curve γ in the metric η is defined to be

$$\int_\gamma d\eta = \sum_{v\in\gamma} \eta(v). \tag{4.2}$$

Let Γ be a collection of vertex curves in G. A vertex metric η is called Γ-*admissible*, if $\int_\gamma d\eta \geqslant 1$ for each $\gamma \in \Gamma$. The *vertex modulus* of Γ is defined by

$$\text{MOD}(\Gamma) = \inf\{\text{area}(\eta) \mid \eta : V \to [0, +\infty) \text{ is } \Gamma\text{-admissible}\}. \tag{4.3}$$

Now, we define the *η-length* of Γ to be the least η-length of a vertex curve in Γ:

$$\text{L}_\eta(\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma d\eta. \tag{4.4}$$

Finally, the *vertex extremal length* of Γ is defined as

$$\text{VEL}(\Gamma) = \sup_{\eta\in\mathscr{M}(G)} \left\{\frac{\text{L}_\eta(\Gamma)^2}{\text{area}(\eta)}\right\}. \tag{4.5}$$

We make the convention that the vertex extremal length is $+\infty$ if Γ is void.

Proposition 4.1.

$$\text{VEL}(\Gamma) = \frac{1}{\text{MOD}(\Gamma)}, \tag{4.6}$$

$$\mathrm{VEL}(\Gamma) = \sup_{\eta\in\mathscr{M}_1}\{\mathrm{area}(\eta)\},\ \textit{where}\ \mathscr{M}_1 = \left\{\eta\in\mathscr{M}(G)\ \middle|\ \inf_{\gamma\in\Gamma}\int_\gamma d\eta = \mathrm{area}(\eta)\right\}. \tag{4.7}$$

Proof. It is clear that if η is Γ-admissible, then

$$\frac{\mathrm{L}_\eta(\Gamma)^2}{\mathrm{area}(\eta)} \geqslant \frac{1}{\mathrm{area}(\eta)}. \tag{4.8}$$

From this inequality, we deduce

$$\begin{aligned}\mathrm{VEL}(\Gamma) &= \sup_{\eta\in\mathscr{M}(G)}\left\{\frac{\mathrm{L}_\eta(\Gamma)^2}{\mathrm{area}(\eta)}\right\} \geqslant \sup\left\{\frac{\mathrm{L}_\eta(\Gamma)^2}{\mathrm{area}(\eta)}\ \middle|\ \eta \text{ is } \Gamma\text{-admissible}\right\}\\ &\geqslant \frac{1}{\inf\{\mathrm{area}(\eta)\mid \eta \text{ is } \Gamma\text{-admissible}\}} = \frac{1}{\mathrm{MOD}(\Gamma)}.\end{aligned}$$

On the other hand, for every vertex metric $\eta\in\mathscr{M}(G)$, we set $\eta_1 = \eta/\mathrm{L}_\eta(\Gamma)$ if $\mathrm{L}_\eta(\Gamma) > 0$. Then η_1 is Γ-admissible and we have

$$\frac{L_\eta(\Gamma)^2}{\mathrm{area}(\eta)} = \frac{L_{\eta_1}(\Gamma)^2}{\mathrm{area}(\eta_1)} = \frac{1}{\mathrm{area}(\eta_1)} \leqslant \frac{1}{\mathrm{MOD}(\Gamma)}. \tag{4.9}$$

If $\mathrm{L}_\eta(\Gamma) = 0$, it is clear that the inequality (4.9) still holds. So $\mathrm{VEL}(\Gamma) \leqslant 1/\mathrm{MOD}(\Gamma)$ and the equation (4.6) is obtained. Noting the fact: every metric in $\mathscr{M}(G)$ will become a metric in $\mathscr{M}_1$ by multiplying an appropriate constant (depending on η), the equation (4.7) can be easily obtained. Here, the verification of the equation (4.7) is left to the reader.

A *path* in $G = (V, E)$ is a finite or infinite sequence $(v_0, v_1, \ldots)$ of vertices such that $[v_i, v_{i+1}] \in E$ for every $i = 0, 1, \ldots$ The set of vertices in a path is a vertex curve, and for the purpose of defining vertex extremal length, we will identify the path with the curve. We say a path $\gamma = (v_0, v_1, \ldots)$ is a *Hamiltonian path* (or *simple path*) if the vertices $v_0, v_1, \ldots$ are distinct. The vertices of γ are denoted by $V(\gamma) = \{v_0, v_1, \ldots\}$. Likewise, for Γ a set of paths in G, we set $V(\Gamma) = \{V(\gamma) : \gamma \in \Gamma\}$. Let V_1, V_2 be nonvoid subsets of vertices. Let $\Gamma_G(V_1, V_2)$ be the set of paths in G joining V_1 and V_2. The *vertex extremal length* between V_1 and V_2 in G is then defined to be

$$\mathrm{VEL}_G(V_1, V_2) = \mathrm{VEL}(\Gamma_G(V_1, V_2)). \tag{4.10}$$

For any three nonvoid subsets of vertices V_1, V_2 and V_3 in the graph G, V_2 is said to *separate* V_1 and V_3, if any path from V_1 to V_3 passes through a vertex in V_2. Note that we *do not* require that $V_i \cap V_j = \emptyset$ for $1 \leqslant i < j \leqslant 3$. In graph theory, V_2 is also called a *cutset* between V_1 and V_3.

Lemma 4.2. *Let $V_1, V_2, \ldots, V_{2m}$ be mutually disjoint, nonvoid subsets of vertices such that for $i_1 < i_2 < i_3$, V_{i_2} separates V_{i_1} from V_{i_3}. Then we have*

$$\mathrm{VEL}_G(V_1, V_{2m}) \geqslant \sum_{k=1}^{m} \mathrm{VEL}_G(V_{2k-1}, V_{2k}). \tag{4.11}$$

Proof. It is clear that we only need to prove the case of $m = 2$. We set

$$\begin{aligned}\Gamma_{1,2} = \Big\{&\gamma = (v_1, v_2, \ldots, v_n) \in \Gamma_G(V_1, V_2) \mid \gamma \text{ is the Hamiltonian path}\\ &\text{and for } 2 \leqslant j \leqslant n-1, v_j \notin \bigcup_{i=1}^{4} V_i\Big\};\end{aligned}$$

$$\begin{aligned}\Gamma_{3,4} = \Big\{&\gamma = (v_1, v_2, \ldots, v_n) \in \Gamma_G(V_3, V_4) \mid \gamma \text{ is the Hamiltonian path}\\ &\text{and for } 2 \leqslant j \leqslant n-1, v_j \notin \bigcup_{i=1}^{4} V_i\Big\}.\end{aligned}$$

From the definitions of the above and the conditions of Lemma 4.2, it is to see $V(\Gamma_{1,2}) \cap V(\Gamma_{3,4}) = \emptyset$. If $V(\Gamma_{1,2})$ (or $V(\Gamma_{3,4})$) is void, then $\Gamma_G(V_1, V_4)$ must be empty set. According to the note of the vertex extremal length, $\mathrm{VEL}_G(V_1, V_4)$ equal to ∞ and the equation (4.11) follows immediately. So we assume that $V(\Gamma_{1,2})$ and $V(\Gamma_{3,4})$ are nonvoid. Now, we consider the subgraph $G_{1,2}$ (or $G_{3,4}$) consisting of the vertices $V(\Gamma_{1,2})$ (or $V(\Gamma_{3,4})$) and the corresponding edges. We claim

$$\mathrm{VEL}_G(V_1, V_2) = \mathrm{VEL}_{G_{1,2}}(V_1, V_2); \tag{4.12}$$

$$\mathrm{VEL}_G(V_3, V_4) = \mathrm{VEL}_{G_{3,4}}(V_3, V_4). \tag{4.13}$$

Because the proof of above two equations is similar, we only prove the equation (4.12). For every metric $\eta \in \mathscr{M}(G)$, $\eta|G_{1,2}$ denote the metric η restrict on $V(G_{1,2})$. In the following, we distinguish two cases:

Case 1. $\mathrm{area}(\eta|G_{1,2}) = 0$. For each path $\gamma \in \Gamma_G(V_1, V_2)$, there exists a Hamiltonian path $\gamma^* \in \Gamma_{1,2} \subseteq \Gamma_{G_{1,2}}(V_1, V_2)$ such that $V(\gamma^*) \subseteq V(\gamma)$. So

$$0 = \int_{\gamma^*} d(\eta|G_{1,2}) = \int_{\gamma^*} d\eta.$$

Since $\gamma^* \in \Gamma_G(V_1, V_2)$, then $\inf_{\gamma\in\Gamma_G(V_1,V_2)} \int_\gamma d\eta = 0$. Hence

$$0 = \frac{(\inf_{\gamma\in\Gamma_G(V_1,V_2)} \int_\gamma d\eta)^2}{\mathrm{area}(\eta)} \leqslant \mathrm{VEL}_{G_{1,2}}(V_1, V_2).$$

Case 2. $\mathrm{area}(\eta|G_{1,2}) > 0$. Clearly $\Gamma_{G_{1,2}}(V_1, V_2) \subseteq \Gamma_G(V_1, V_2)$. For each path γ^* in $\Gamma_{G_{1,2}}(V_1, V_2)$, we have

$$\inf_{\gamma\in\Gamma_G(V_1,V_2)} \int_\gamma d\eta \leqslant \int_{\gamma^*} d\eta = \int_{\gamma^*} d(\eta|G_{1,2}).$$

When γ^* runs out of $\Gamma_{G_{1,2}}(V_1, V_2)$, noting that $\eta|G_{1,2} \in \mathscr{M}(G_{1,2})$, we deduce

$$\frac{(\inf_{\gamma\in\Gamma_G(V_1,V_2)} \int_\gamma d\eta)^2}{\mathrm{area}(\eta)} \leqslant \frac{(\inf_{\gamma^*\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\gamma^*} d(\eta|G_{1,2}))^2}{\mathrm{area}(\eta|G_{1,2})} \leqslant \mathrm{VEL}_{G_{1,2}}(V_1, V_2).$$

Combining the two cases, noting that η being an arbitrary metric in $\mathscr{M}(G)$, we obtain $\mathrm{VEL}_G(V_1, V_2) \leqslant \mathrm{VEL}_{G_{1,2}}(V_1, V_2)$.

On the other hand, for every $\eta^* \in \mathscr{M}(G_{1,2})$, we define

$$\eta(v) = \begin{cases} \eta^*(v), & v \in V(G_{1,2}), \\ 0, & \text{otherwise}. \end{cases}$$

For each path $\gamma \in \Gamma_G(V_1, V_2)$, there exists a path $\gamma^* \in \Gamma_{G_{1,2}}(V_1, V_2)$ with $V(\gamma^*) \subseteq V(\gamma)$. So $\int_\gamma d\eta \geqslant \int_{\gamma^*} d\eta = \int_{\gamma^*} d\eta^* \geqslant \inf_{\tilde\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\tilde\gamma} d\eta^*$. Hence

$$\inf_{\tilde\gamma\in\Gamma_G(V_1,V_2)} \int_{\tilde\gamma} d\eta \geqslant \inf_{\tilde\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\tilde\gamma} d\eta^*.$$

Meanwhile, the inequality

$$\inf_{\tilde\gamma\in\Gamma_G(V_1,V_2)} \int_{\tilde\gamma} d\eta \leqslant \inf_{\tilde\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\tilde\gamma} d\eta^*$$

is clear. So

$$\inf_{\tilde\gamma\in\Gamma_G(V_1,V_2)} \int_{\tilde\gamma} d\eta = \inf_{\tilde\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\tilde\gamma} d\eta^*.$$

Noting that $\mathrm{area}(\eta^*) = \mathrm{area}(\eta)$, then we get

$$\mathrm{VEL}_G(V_1, V_2) \geqslant \frac{\inf_{\tilde\gamma\in\Gamma_G(V_1,V_2)} \int_{\tilde\gamma} d\eta}{\mathrm{area}(\eta)} = \frac{\inf_{\tilde\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_{\tilde\gamma} d\eta^*}{\mathrm{area}(\eta^*)}.$$

Because η^* is an arbitrary metric in $\mathscr{M}(G_{1,2})$, then $\mathrm{VEL}_G(V_1,V_2) \geqslant \mathrm{VEL}_{G_{1,2}}(V_1,V_2)$, the claim is proved.

Now, it is time to prove the inequality (4.11). We choose the metrics $\eta_1 \in \mathscr{M}_1(G_{1,2})$ and $\eta_2 \in \mathscr{M}_1(G_{3,4})$, that is,

$$\inf_{\gamma\in\Gamma_{G_{1,2}}(V_1,V_2)} \int_\gamma d\eta_1 = \mathrm{area}(\eta_1) \text{ and } \inf_{\gamma\in\Gamma_{G_{3,4}}(V_3,V_4)} \int_\gamma d\eta_2 = \mathrm{area}(\eta_2).$$

Since $V(G_{1,2}) \cap V(G_{3,4}) = \emptyset$, then we can define the metric $\eta \in \mathscr{M}(G)$ as follows:

$$\eta(v) = \begin{cases} \eta_1(v), & v \in V(G_{1,2}), \\ \eta_2(v), & v \in V(G_{3,4}), \\ 0, & \text{otherwise.} \end{cases}$$

For each path $\gamma \in \Gamma_G(V_1,V_4)$, since V_{i_2} separates V_{i_1} from V_{i_3} when $i_1 < i_2 < i_3$, then there exist two paths $\gamma_1 \in \Gamma_{G_{1,2}}(V_1,V_2)$ and $\gamma_2 \in \Gamma_{G_{3,4}}(V_3,V_4)$ with $V(\gamma_1)\cap V(\gamma_2)$ and $V(\gamma_1)\cup V(\gamma_2) \subseteq V(\gamma)$. So

$$\int_\gamma d\eta \geqslant \int_{\gamma_1} d\eta_1 + \int_{\gamma_2} d\eta_2 \geqslant \mathrm{area}(\eta_1) + \mathrm{area}(\eta_2) = \mathrm{area}(\eta).$$

Hence, we deduce $\mathrm{VEL}_G(V_1,V_4) \geqslant \mathrm{area}(\eta) = \mathrm{area}(\eta_1) + \mathrm{area}(\eta_2)$. It follows that

$$\mathrm{VEL}_G(V_1,V_4) \geqslant \sup_{\eta_1\in\mathscr{M}_1(G_{1,2})} \mathrm{area}(\eta_1) + \sup_{\eta_2\in\mathscr{M}_1(G_{3,4})} \mathrm{area}(\eta_2).$$

Combining the equation (4.7) in Proposition 4.1 and the claim, Lemma 4.2 is proved.

The following property of vertex extremal length is called Duality Theorem in [5]. For the completeness, here we give a proof.

Proposition 4.3. *Let $G = (V,E)$ be a finite connected graph, and let V_1 and V_2 be nonvoid subsets of vertices in G. Let $\Gamma^*_G(V_1,V_2)$ be the collection of vertex curves in G which separate V_1 from V_2, then*

$$\mathrm{VEL}(\Gamma^*_G(V_1,V_2)) = \frac{1}{\mathrm{VEL}(\Gamma_G(V_1,V_2))}. \tag{4.14}$$

Proof. It is clear that $\mathrm{VEL}(\Gamma_G(V_1,V_2)) > 0$. For the convenience, we let Γ (or Γ^*) denote $\Gamma_G(V_1,V_2)$ (or $\Gamma^*_G(V_1,V_2)$). An *extremal metric* for $\mathrm{VEL}(\Gamma)$ is one which realize the supremum of the $\mathrm{VEL}(\Gamma)$. In [12] (see Lemma 3.1), we know that there exist an extremal metric η for $\mathrm{VEL}(\Gamma)$ with $\mathrm{area}(\eta) = 1$. Let γ^* be a curve of least η-length in Γ^* (because G is a finite graph), that is, $\int_{\gamma^*} d\eta = \inf_{\tilde\gamma\in\Gamma^*}\int_{\tilde\gamma} d\eta$. For $t \geqslant 0$, set

$$\eta_t(v) = \begin{cases} \eta(v)+t, & v \in V(\gamma^*), \\ \eta, & \text{otherwise.} \end{cases}$$

Since every path in Γ must insect γ^*, we have $L_{\eta_t} \geqslant L_\eta + t$, where $L_\eta = L_\eta(\Gamma) = \inf_{\gamma\in\Gamma}\int_\gamma d\eta$. So $D_+(L_{\eta_t}) \geqslant 1$, where D_+ denotes the one sided Dini's derivative with respect to t, as $t \downarrow 0$. It is also easy to compute $D_+(\mathrm{area}(\eta_t))$:

$$D_+(\mathrm{area}(\eta_t)) = \sum_{v\in V(\gamma^*)} D_+((\eta(v)+t)^2) = \sum_{v\in V(\gamma^*)} 2\eta(v) = 2\int_{\gamma^*} d\eta.$$

Let $\hat L(\eta) = L_\eta^2/\mathrm{area}(\eta)$. Since η is the extremal metric (i.e., $\hat L(\eta) \geqslant \hat L(\eta_t)$), we have

$$0 \geqslant D_+(\hat L(\eta_t)) = D_+\left(\frac{L_{\eta_t}^2}{\mathrm{area}(\eta_t)}\right) \geqslant \frac{\mathrm{area}(\eta)^2 D_+(L_{\eta_t}^2) - L_\eta^2 D_+(\mathrm{area}(\eta_t)^2)}{\mathrm{area}(\eta)^2}.$$

Using our previous computations, and the normalization $\mathrm{area}(\eta) = 1$, we have

$$0 \geqslant D_+(L_{\eta_t}^2) - L_\eta^2 D_+(\mathrm{area}(\eta_t)) \geqslant 2L_\eta - 2L_\eta^2 \int_{\gamma^*} d\eta.$$

Since $\int_{\gamma^*} d\eta = \inf_{\tilde{\gamma}\in\Gamma^*} \int_{\tilde{\gamma}} d\eta$ and η is an extremal metric for VEL(Γ) with area(η) = 1, then

$$\frac{(\inf_{\tilde{\gamma}\in\Gamma^*} \int_{\tilde{\gamma}} d\eta)^2}{\text{area}(\eta)} \geqslant \frac{\text{area}(\eta)}{(L_\eta)^2} = \frac{1}{\text{VEL}(\Gamma)},$$

this gives $\text{VEL}(\Gamma^*) \geqslant 1/\text{VEL}(\Gamma)$. With the same proof, noting that the extremal metric for $\text{VEL}(\Gamma^*)$ is still exist, we can get $\text{VEL}(\Gamma^*) \leqslant 1/\text{VEL}(\Gamma)$. So Proposition 4.3 is proved.

We recall that $c(r)$ denote the circle of radius r centered at 0. Let P be a disk packing in $\mathbb{C}$, we set $V_c = V_c(P) = \{v \mid P(v) \cap c \neq \emptyset\}$. With these symbols, we have the following results (the disk patterns case please refer to [4]).

Lemma 4.4. *Let $G = (V, E)$ be a graph. Suppose that P is a disk packing in $\mathbb{C}$ which realize G. Then for any $r_2 > r_1 > 0$,*

$$\text{VEL}(V_{c(r_1)}, V_{c(r_2)}) \geqslant \frac{(r_2 - r_1)^2}{(8 + (2\pi)^2)(r_2)^2}. \tag{4.15}$$

In particular, if $r_2 \geqslant 2r_1$, then,

$$\text{VEL}(V_{c(r_1)}, V_{c(r_2)}) \geqslant \frac{1}{(32 + (4\pi)^2)}. \tag{4.16}$$

Proof. Let $R(r_1, r_2)$ denote the ring domain $\{z \mid r_1 < |z| < r_2\}$. For a vertex $v \in V$, if the disk $P(v) \cap R(r_1, r_2) \neq \emptyset$, let $l(v) = \max\{r_1, \text{dist}(0, P_v)\}$, $L(v) = \min(r_2, \sup_{z\in P(v)} |z|)$. Define $d(v)$ as follows (see Figure 3):

$$d(v) = \begin{cases} L(v) - l(v), & \text{when } P(v) \cap R(r_1, r_2) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

From the geometric signification of $d(v)$, it is easy to see that

$$\pi\left(\frac{d(v)}{2}\right)^2 \leqslant \text{area}(P(v)). \tag{4.17}$$

Let γ be a path in G joining $V_{c(r_1)}$ and $V_{c(r_2)}$. Then $P(\gamma) = \bigcup_{v\in V(\gamma)} P(v)$ is a continuum joining $c(r_1)$ and $c(r_2)$(see Figure 4).

It follows that $\sum_{v\in V(\gamma)} d(v) \geqslant r_2 - r_1$. Let $D(r_2)$ be a close disk with radius r_2 centered at 0 and $\eta(v) = d(v)/(r_2 - r_1)$. Thus η is $\Gamma_G(V_{c(r_1)}, V_{c(r_2)})$-admissible, and then by (4.3) and (4.17),

$$\begin{aligned}\text{MOD}(\Gamma_G(V_{c(r_1)}, V_{c(r_2)})) \leqslant \text{area}(\eta) \leqslant & \sum_{\text{area}(P(v)\cap D(r_2))\geqslant(1/2)\text{area}(P(v))} \frac{4}{\pi}\frac{\text{area}(P(v))}{(r_2 - r_1)^2} \\ & + \sum_{\text{area}(P(v)\cap D(r_2))<(1/2)\text{area}(P(v))} \frac{d(v)^2}{(r_2 - r_1)^2}.\end{aligned}$$

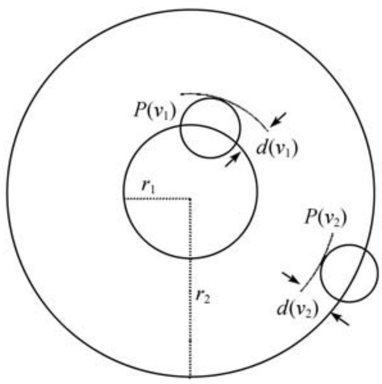

Figure 3 The geometric signification of $d(v)$

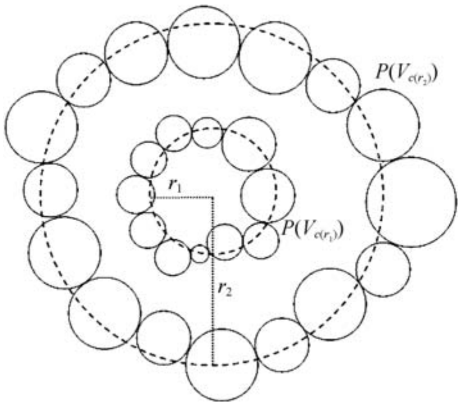

Figure 4 For any γ in G joining $V_{c(r_1)}$ and $V_{c(r_2)}$ is a continuum joining $c(r_1)$ and $c(r_2)$

Since any point in $\mathbb{C}$ lies in at most one disk in the disk packing P, the first sum on the right-hand side is bounded by $(4/\pi)\cdot 2\cdot \text{area}(D(r_2))/(r_2-r_1)^2 = 8(r_2)^2/(r_2-r_1)^2$. The second sum is bounded by $(2\pi r_2)^2/(r_2-r_1)^2$, because,

$$\sum_{\text{area}(P_v\cap D(r_2))<(1/2)\text{area}(P_v)} d(v) \leqslant \text{length}(\partial D(r_2)) = 2\pi r_2. \tag{4.18}$$

As a consequence,

$$\text{MOD}(\Gamma_G(V_{c(r_1)}), V_{c(r_2)}) \leqslant \text{area}(\eta) \leqslant [8+(2\pi)^2]\frac{(r_2)^2}{(r_2-r_1)^2}. \tag{4.19}$$

This implies (4.15).

5 The proof of Theorem 1.2

Let G be a graph and P be a circle packing in $\mathbb{C}$ which realizes G. If G is a closed disk triangulation graph, then for any circle c with $c \subseteq \text{supp}(P)$, $V_c = V_c(P)$ is a connected set of vertices, and thus $P(V_c) = \sum_{v\in V_c} P(v)$ is pathwise connected. It is also easy to see that for any path $\gamma = (v_0, v_2, \ldots, v_l)$, the set $P(\gamma) = \sum_{k=0}^{l} P(v_k)$ is pathwise connected. The following lemma for disk patterns case appeared in [10], for completeness, here we give a proof.

Lemma 5.1. *Let $G=(V,E)$ be a closed disk triangulation graph and P be a circle packing in $\mathbb{C}$ which realizes G. Let $V_0=\{v_0\}$, V_1, V_2 and V_3 be mutually disjoint, finite, connected subsets of interior vertices, and let $V_4 = \partial V$. Assume that for any $0 \leqslant i_1 < i_2 < i_3 \leqslant 4$, the set V_{i_2} separates V_{i_1} from V_{i_3}. If*

$$\text{VEL}(V_1, V_2) > 72 + (6\pi)^2, \tag{5.1}$$

then there is some $r>0$ such that for any circle c concentric with $P(v_0)$ and radius $\rho(c) \in [r, 2r]$, the vertex set $V_c(P)$ separates V_1 from V_2.

Proof. We may assume that the disk $P(v_0)$ is centered at 0. Let $r = \sup\{\text{dist}(0,z) : z \in P(v) \text{ and } v \in V_1\}$, where $\text{dist}(\cdot,\cdot)$ denotes the euclidean distance. Without loss of generality, we may assume that $r=1$.

Let $\gamma^* \in \Gamma_G^*(V_1, V_2)$. Then γ^* also separates $V_1 \cup V_0$ from ∂V. Thus by the assumptions on V_1, V_2 and G, we have

$$\text{diam}(P(\gamma^*)) \geqslant r = 1. \tag{5.2}$$

By contradiction, let us assume that Lemma 5.1 fails. Then there is a circle $c(\rho_1)$ centered at $z_0=0$ and of radius $\rho_1 \in [1,2]$ such that $V_{c(\rho_1)}$ does not separate V_1 from V_2. So there is a path γ_0 in the graph joining V_1 and V_2 such that $V(\gamma_0) \cap V_{c(\rho_1)} = \emptyset$. That is, $P(\gamma_0) \cap c(\rho_1) = \emptyset$. In other words, the continuum $P(\gamma_0)$ is either contained in the interior of the disk $D(\rho_1)$ or in the exterior of $D(\rho_1)$. But since $D(1)$ has nonempty intersection with every $P(v)$, $v \in V_1$, and since γ_0 contains a vertex of V_1, we have $P(\gamma_0) \cap D(1)) \neq \emptyset$.

Therefore $P(\gamma_0)$ is contained in the interior of $D(\rho_1)$ (recall that $\rho_1 \geqslant 1$). As $\gamma_0 \in \Gamma_G(V_1, V_2)$ and $\gamma^* \in \Gamma^*_G(V_1, V_2)$, we deduce that

$$P(\gamma^*) \cap D(\rho_1) \supseteq P(\gamma^*) \cap P(\gamma_0) \supseteq P(V(\gamma^*) \cap V(\gamma_0)) \neq \emptyset. \tag{5.3}$$

For any vertex v, let $\eta(v) = \mathrm{diam}(P(v) \cap D(3))$. Then $\eta : V \to [0, \infty)$ is a vertex metric in the graph. If $\gamma^* \in \Gamma^*_G(V_1, V_2)$ is connected, then by (5.3) it follows that $P(\gamma^*)$ is either contained in $D(3)$, or is a continuum joining $c(\rho_1)$ and $c(3)$. In either case, by (5.2) and the inequality $3 \geqslant \rho_1 + 1$, we have $\int_{\gamma^*} d\eta = \sum_{v \in V(\gamma^*)} \mathrm{diam}(P(v) \cap D(3)) \geqslant 1$. Again by the connected cut lemma in [5] (or by the Alexander's Lemma in [9]), it is easy to see that each vertex curve in $\Gamma^*_G(V_1, V_2)$ contains a connected vertex subcurve γ^* in $\Gamma^*_G(V_1, V_2)$. It follows that η is $\Gamma^*_G(V_1, V_2)$-admissible. Thus,

$$\mathrm{MOD}(\Gamma^*_G(V_1, V_2)) \leqslant \mathrm{area}(\eta).$$

By an argument as in Lemma 4.4 (see (4.19)), we deduce that

$$\mathrm{MOD}(\Gamma^*_G(V_1, V_2)) \leqslant \mathrm{area}(\eta) \leqslant [8 + (2\pi)^2] \cdot 3^2.$$

By Propositions 4.1 and 4.3, this contradicts (5.1).

Proof of Theorem 1.2. Let $K \subset D$ be a compact subset and for each n let v^n_0, v^n_1 be two vertices with $P^n(v^n_0) \cap K \neq \emptyset$ and v^n_1 neighboring to v^n_0. Suppose that $d = \mathrm{dist}(K, \partial D) > 0$. We may assume that the disk $P^n(v_0)$ and the corresponding disk $\tilde{P}^n(v_0)$ all center at 0.

Let k be a fixed integer:

$$k = 2 + \text{the integer part of } [(72 + (6\pi)^2)(32 + (4\pi)^2)]. \tag{5.4}$$

Let n be sufficient large such that $n > k$ and $\ln \frac{d}{2\delta_n} / \ln 4 > 2k$. For each such as n, let $r_i = 4^j \delta_n$, and let $V_0 = \{v^n_0, v^n_1\}$, $V_i = V_{c(r_i)}$ for $1 \leqslant j \leqslant 2k$. By the definition of δ_n, it follows that no disk in P^n can intersect both $c(r_i)$ and $c(r_j)$ for $i \neq j$. In other words, $V_{c(r_i)} \cap V_{c(r_j)} = \emptyset$. Moreover, $V_{c(r_i)} \cap N(V_{c(r_j)}) = \emptyset$ and $N(V_{c(r_i)}) \cap V_{c(r_j)} = \emptyset$ where $N(V)$ denotes the set of vertices that are in V or neighbor with some vertex in V. Since $\ln \frac{d}{2\delta_n} / \ln 4 > 2k$, then for sufficient large n, by Lemma 2.1, $c(r_{2k}) \subset \mathrm{supp}(P^n)$. Let $V_{2k+1} = \partial V(P^n)$. For any $0 \leqslant i_1 < i_2 < i_3$, it is clear that V_{i_2} separates V_{i_1} from V_{i_3}. Therefore, by Lemma 4.2,

$$\mathrm{VEL}(V_2, V_{2k-1}) \geqslant \sum_{i=1}^{k-1} \mathrm{VEL}(V_{2i}, V_{2i+1}).$$

By Lemma 4.4, $\mathrm{VEL}(V_{2i}, V_{2i+1}) \geqslant 1/(32 + (4\pi)^2)$. It follows that,

$$\mathrm{VEL}(V_2, V_{2k-1}) \geqslant \frac{k-1}{32 + (4\pi)^2}.$$

According to the definition of the integer k, then

$$\mathrm{VEL}(V_2, V_{2k-1}) > 72 + 6\pi^2. \tag{5.5}$$

Applying Lemma 5.1 to the packing $Q = \tilde{P}^n$, there is some r^*, such that for any $\rho \in [r^*, 2r^*]$, the vertex set $V^*_{c(\rho)}$ ($V^*_{c(\rho)} = \{v \mid Q(v) \cap c(\rho) \neq \emptyset\}$) separates V_2 from V_{2k-1}. It follows that $V^*_{c(\rho)}$ also separates V_1 from V_{2k} and separates V_0 from V_{2k+1}. Moreover, $V^*_{c(\rho)} \cap V_1 \subseteq V_2 \cap V_1 = \emptyset$ and $V^*_{c(\rho)} \cap V_{2k} \subseteq N(V_{2k-1}) \cap V_{2k} = \emptyset$. Therefore $\tilde{P}^n(V_1)$ is contained in the interior disk of $c(r^*)$, and $\tilde{P}^n(V_{2k})$ lies in the exterior disk of $c(2r^*)$ (see Figure 5).

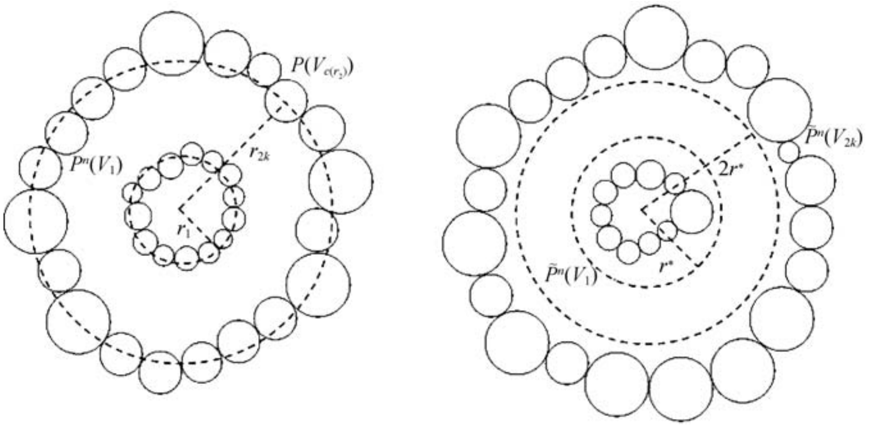

Figure 5 The sets $P^n(V_1)$, $P^n(V_{2k})$, $\tilde{P}^n(V_1)$ and $\tilde{P}^n(V_{2k})$

Consider the subgraph G_1 of $G(P^n)$ consisting of the vertices v for which $P^n(v) \cap D(r_1) \neq \emptyset$ and the corresponding edges. Then the disks $\tilde{P}^n(v), v \in V(G_1)$, are contained in the disk $D(r^*) \subset D(2r^*)$. Since $D(r_1) \subseteq \mathrm{supp}(P^n)$ (recall that $c(r_{2k}) \subset \mathrm{supp}(P^n)$), then for each $v \in \partial V(G_1)$, we have $P^n(v) \cap c(r_1) \neq \emptyset$. So the inequality $\rho_{\rm hyp}((1/(2r^*))\tilde{P}^n(v)) \leqslant \rho_{\rm hyp}((1/r_1)P^n(v))$ is true for each boundary vertex v in $V(G_1)$. Using Lemma 3.4, we deduce that for each $v \in V(G_1)$,

$$\rho_{\rm hyp}\left(\frac{1}{2r^*}\tilde{P}^n(v)\right) \leqslant \rho_{\rm hyp}\left(\frac{1}{r_1}P^n(v)\right).$$

In particular, for $v = v_j^n, j = 0, 1$,

$$\rho_{\rm hyp}\left(\frac{1}{2r^*}\tilde{P}^n(v_j^n)\right) \leqslant \rho_{\rm hyp}\left(\frac{1}{r_1}P^n(v_j^n)\right). \tag{5.6}$$

On the other hand, consider the subgraph G_2 consisting of the vertices v for which $\tilde{P}^n(v) \cap D(2r^*) \neq \emptyset$ and the corresponding edges. Then the disks $P^n(v), v \in V(G_2)$, are contained in the interior of $D(r_{2k})$. It follows by Lemma 3.4 again that

$$\rho_{\rm hyp}\left(\frac{1}{2r^*}\tilde{P}^n(v)\right) \geqslant \rho_{\rm hyp}\left(\frac{1}{r_{2k}}P^n(v)\right), \quad \text{for all } v \in V(G_2).$$

Thus, for $v = v_j^n, j = 0, 1$,

$$\rho_{\rm hyp}\left(\frac{1}{2r^*}\tilde{P}^n(v_j^n)\right) \geqslant \rho_{\rm hyp}\left(\frac{1}{r_{2k}}P^n(v_j^n)\right). \tag{5.7}$$

Since the disks $(1/(2r^*))\tilde{P}^n(v_j^n)$, $(1/r_1)P^n(v_j^n)$, $(1/r_{2k})P^n(v_j^n)$, $j = 0, 1$, are all contained in $(1/2)U$, it follows that their radii are comparable with their hyperbolic radii. So by (5.6) and (5.7), there is a universal constant $C_0 > 0$ such that

$$\frac{1}{2r^*}\rho(\tilde{P}^n(v_j^n)) \leqslant \frac{1}{r_1}\rho(P^n(v_j^n)),$$

and

$$\frac{1}{2r^*}\rho(\tilde{P}^n(v_j^n)) \geqslant \frac{1}{r_{2k}}\rho(P^n(v_j^n)) = 4^{-(2k-1)}\frac{1}{r_1}\rho(P^n(v_j^n)),$$

where $j = 0, 1$ (recall that $r_{2k} = 4^{2k-1}r_1$). By a simple computation, we have

$$\frac{\rho(\tilde{P}^n(v_1^n))}{\rho(P^n(v_1^n))} \leqslant ((C_0)^2 4^{2k-1})\frac{\rho(\tilde{P}^n(v_0^n))}{\rho(P^n(v_0^n))} = C\frac{\rho(\tilde{P}^n(v_0^n))}{\rho(P^n(v_0^n))},$$

and

$$\frac{\rho(\tilde{P}^n(v_1^n))}{\rho(P^n(v_1^n))} \geqslant \frac{1}{(C_0)^2 4^{2k-1}}\frac{\rho(\tilde{P}^n(v_0^n))}{\rho(P^n(v_0^n))} = \frac{1}{C}\frac{\rho(\tilde{P}^n(v_0^n))}{\rho(P^n(v_0^n))}.$$

So Theorem 1.2 is proved.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 10701084, 10501046) and the Natural Science Foundation of Chongqing, China (Grant No. CSTC2008BB0151). We cordially thank Prof. He Zheng-xu for his instructive help and stimulating discussion.

References

1 Brightwell G R, Scheinerman E R. Representations of planar graphs. SIAM J Discrete Math, 1993, 6: 214–229
2 Cannon J. The combinatorial Riemann mapping theorem. Acta Math, 1994, 173: 155–234
3 De Verdiére Y C. Un principe variationnel pour les empilements de cercles. Invent Math, 1991, 104: 655–669
4 He Z H. Rigidity of infinite disk patterns. Ann of Math, 1999, 149: 1–33
5 He Z H, Schramm O. Hyperbolic and parabolic packings. Discrete Comput Geom, 1995, 14: 123–149
6 He Z H, Schramm O. On the convergence of circle packings to the Riemann map. Invent Math, 1996, 125: 285–305
7 Koebe P. Kontaktprobleme der konformen Abbildung. Ber Sächs Akad Wiss Leizig Math Phys Kl, 1936, 88: 141–164
8 Marden A, Rodin B. On Thurston's formulation and proof of Andreev's theorem. In: Lecture Notes in Mathematics, Vol. 1435. Berlin: Springer, 1989, 103–164
9 Newman M H A. Elements of the Topology of Plane Sets of Points. Cambridge: Cambridge University Press, 1954
10 Rodin B. Schwarz's lemma for circle packings. Invent Math, 1987, 89: 271–289
11 Rodin B, Sullivan D. The convergence of circle packing to the Riemann mapping. J Differential Geom, 1987, 26: 349–360
12 Schramm O. Square tilings with prescribed combinatorics. Israel J Math, 1993, 84: 97–118
13 Schramm O. Transboundary extremal length. J d'Analyse Math, 1995, 66: 307–329
14 Schramm O. Conformal uniformization and packings. Israel J Math, 1996, 93: 399–428
15 Thurston W. The finite Riemann mapping theorem. Invited talk, International Symposium in Celebraition of the Proof of the Bieberbach conjecture, Purdue University, 1985