QUASIHYPERBOLIC METRIC AND QUASISYMMETRIC MAPPINGS IN METRIC SPACES

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Abstract. In this paper, we prove that the quasihyperbolic metrics are quasi-invariant under a quasisymmetric mapping between two suitable metric spaces. Meanwhile, we also show that quasi-invariance of the quasihyperbolic metrics implies that the corresponding map is quasiconformal. At the end of this paper, as an application of above theorems, we prove that the composition of two quasisymmetric mappings in metric spaces is a quasiconformal mapping.

1. Introduction

During the past few decades, modern geometric function theory of quasisymmetric and quasiconformal mappings has been studied from several points of view. Quasisymmetric mappings on the real line were first introduced by Beurling and Ahlfors [1]. They found a way to extend each quasisymmetric self-mapping of the real line to a quasiconformal self-mapping of the upper half-planes. This concept was later promoted by Tukia and Väisälä [11], who introduced and studied quasisymmetric mappings between arbitrary metric spaces. In 1990, based on the idea of quasisymmetry, Väisälä developed a "dimension-free" theory of quasiconformal mappings in infinite-dimensional Banach spaces. See also [18, 19, 20, 21, 22]. In 1998, Heinonen and Koskela [8] showed that these concepts, quasiconformality and quasisymmetry, are quantitatively equivalent in a large class of metric spaces, which includes Euclidean spaces. Since these two concepts are equivalent, mathematicians show much interest in the research of quasisymmetric mappings between suitable metric spaces.

Following analogous notations and terminologies of [7, 8, 22, 12], now we give the definitions of quasisymmetry and quasiconformality.

Definition 1.1. Given a homeomorphism $f : X \to Y$ between two metric spaces, f is said to be *quasisymmetric* if there is a constant $H < \infty$, for all $x \in X$ and all $r > 0$,

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)} \le H, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 30C65.

Key words and phrases. Quasisymmetry; Quasiconformality; Quasihyperbolic metry; Quasiconvex metric space.

The first author was supported by the Fundamental Research Funds for the Central Universities Grant No. CQDXWL-2012-013.

The second author was supported by NSFC Grant No. 10831004.

where

$$L_f(x,r) := \sup_{|y-x|\le r} \big\{|f(y)-f(x)|\big\} \tag{1.2}$$

and

$$l_f(x,r) := \inf_{|y-x|\ge r} \big\{|f(y)-f(x)|\big\}. \tag{1.3}$$

Note that here and hereafter we use the distance notation $|x-y|$ in any metric space.

The slightly different formulation used here can be easily turned into the following stronger quasisymmetry condition.

Definition 1.2. A homeomorphism $f : X \to Y$ between two metric spaces is called *η-quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$|x-a| \le t|x-b| \quad \text{implies} \quad |f(x)-f(a)| \le \eta(t)|f(x)-f(b)| \tag{1.4}$$

for each $t > 0$ and for each triple x, a, b of points in X.

Obviously, (1.4) implies quasisymmetry as defined in (1.1). In general, these two notions are not equivalent. However, in any path-wise connected doubling metric spaces we know that (1.1) implies (1.4). Please refer to [17].

Quasiconformal mappings are homeomorphisms that distort the shape of infinitesimal balls by a uniformly bounded amount. This requirement makes sense in every metric space.

Definition 1.3. A homeomorphism f from a metric space X to a metric space Y is said *quasiconformal* if there is a constant $H < \infty$ so that

$$\limsup_{r\to 0} H_f(x,r) \le H \tag{1.5}$$

for all $x \in X$, where $H_f(x,r)$ is defined in (1.1).

In [8], Heinonen and Koskela proved that quasiconformal mappings between Ahlfors $Q(>1)$-regular metric measure spaces are quasisymmetric, provided that the source is a Loewner space and the target space satisfies a quantitative connectivity condition.

Gehring and others [3, 4] introduced the quasihyperbolic metric. It is an important tool in the research of quasisymmetric and quasiconformal mappings between metric spaces. In [3] (Theorem 3), Gehring and Osgood [3] proved that quasihyperbolic metric is quasi-invariant under any K-quasiconformal mappings of a domain $D \subset \mathbb{R}^n$. We wish to point out that the use of the term "quasiconformal" in [3] differs from its use in this paper. Their result can be stated as follows:

Theorem 1.4. *There exists a constant c depending only on n and K with the following property. If f is a K-quasiconformal mapping of domain D onto D', then*

$$k_{D'}\big(f(x_1), f(x_2)\big) \le c\ \max\big(k_D(x_1,x_2), k_D(x_1,x_2)^\alpha\big), \quad \alpha = K^{1/(1-n)},$$

for all $x_1, x_2 \in D$.

Remark 1.5. For any metric space X, a non-empty subset $D \subseteq X$ is said a *domain* if it is open and connected. For the concept of $k_D(\cdot,\cdot)$ and $k_{D'}(\cdot,\cdot)$, please see Definition 2.2 in Section 2.

In 1990, Väisälä studied quasiconformal mappings between infinite-dimensional Banach spaces and obtained a series of novel results. He also obtained an alternative version to Theorem 1.4.

Under suitable geometric conditions (see Section 2), in this paper we shall prove a more general result (Theorem 1.6) for metric spaces. Our proof is based on a refinement of the method due to Väisälä [22].

Theorem 1.6. *Let X be a c-quasiconvex complete metric space and let Y be a c'-quasiconvex metric space. Suppose that $G \subsetneq X$ and $G' \subsetneq Y$ are two domains and f is an H-quasisymmetry from G onto G'. Then there exists a non-decreasing function $\psi : (0, \infty) \to (0, \infty)$ such that, for all $x, y \in G$,*

$$k'\big(f(x), f(y)\big) \leq \psi\big(k(x, y)\big).$$

Note that the function ψ depends only on c, c', H and satisfies $\psi(t) \to 0$ as $t \to 0$.

It is clear that the converse to Theorem 1.6 is also an interesting problem. To study this problem, we introduce the following definition.

Definition 1.7. Let $D \subseteq X$ be a domain in a metric space X. A point $x \in D$ is said to be a *cut point* if $D\backslash\{x\}$ is not connected. A domain D is said to be a *non-cut-point* domain if it has no cut points.

For any two c-convex (see Section 2) and complete metric spaces, we prove that quasi-invariance of the quasihyperbolic metrics implies the corresponding map is quasiconformal.

Theorem 1.8. *Let X be a c-quasiconvex, complete metric space and $G \subsetneq X$ be a non-cut-point domain. Let $G' \subsetneq Y$ be be a domain in a metric space Y. Suppose that $f : G \to G'$ is a homeomorphism. If for any sub-domain $E \subseteq G$ and $\forall x, y \in E$,*

$$k_{E'}\big(f(x), f(y)\big) \leq \varphi\big(k_E(x, y)\big), \tag{1.6}$$

where $E' = f(E)$ and φ is an increasing function, then f is an H-quasiconformal mapping with

$$H = e^{\varphi(2c)} - 1.$$

In the appendix we will give an example to show the non-cut-point assumption indeed can not be ruled out.

As an application of Theorem 1.6 and Theorem 1.8, we show that the composite mapping of two quasisymmetric mappings in a large class of metric spaces is a quasiconformal mapping.

Theorem 1.9. *Let X (resp. Y) be a c_1 (resp. c_2)-quasiconvex and complete metric space and let Z be a c_3-quasiconvex metric space. Suppose that $G \subsetneq X$ is a non-cut-point domain. For any two domains $G' \subsetneq Y$ and $G'' \subsetneq Z$, if $f : G \to G'$ is an H_1-quasisymmetric mapping and $g : G' \to G''$ is an H_2-quasisymmetric mapping, then $g \circ f$ is an H-quasiconformal mapping, where H depends only on the above data.*

In [8], Heinonen and Koskela showed that quasiconformal mappings between metric spaces of "bounded geometry" are quasisymmetric. Their result is as follows:

Theorem 1.10. *Suppose that X and Y are bounded Q-regular metric spaces with $Q > 1$, Furthermore, suppose that X is a Loewner space and Y is linearly locally connected. If f is a quasiconformal map from X onto Y, then f is quasisymmetric.*

Using the same assumptions as in Theorem 1.9, by combing Theorem 1.9 and Theorem 1.10, we shall now have the following corollary.

Corollary 1.11. Suppose that X and Z are bounded Q-regular metric spaces with $Q > 1$. Furthermore, suppose that X is a Loewner space and Z is linearly locally connected. If $f : G \to G'$ and $g : G' \to G''$ are quasisymmetric, then their composition $g \circ f$ is quasisymmetric.

2. Quasihyperbolic metric

Let X be a metric space and let

$$B(x, r) = \{y : |y - x| \le r\}, \quad U(x, r) = \{y : |y - x| < r\}$$

be the closed and open balls with center $x \in X$ and radius $r > 0$. Denote

$$S(x, r) = \{y : |y - x| = r\}.$$

If $B_r = B(x, r)$ (or $U_r = U(x, r)$), then $\lambda B_r = B(x, \lambda r)$ (or $\lambda U_r = U(x, \lambda r)$) for any $\lambda > 0$. The closure of a set A is denoted by $\overline{A}$.

By a *curve* we mean any continuous mapping $\gamma : [a, b] \to X$. The *length* of γ is defined by

$$l(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i+1})| \right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. The curve is *rectifiable* if $l(\gamma) < \infty$.

The *length function* associated with a rectifiable curve $\gamma : [a, b] \to X$ is $s_\gamma : [a, b] \to [0, l(\gamma)]$, given by $s_\gamma(t) = l(\gamma|_{[a,t]})$. For any rectifiable curve $\gamma : [a, b] \to X$, there is a unique curve $\gamma_s : [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. Moreover, $l\big(\gamma_s|_{[0,t]}\big) = t$ for every $t \in [0, l(\gamma)]$. The curve γ_s is called the *arc length parametrization* of γ.

If γ is a rectifiable curve in X, the line integral over γ of each nonnegative Borel function $\varrho : X \to [0, \infty]$ is

$$\int_\gamma \varrho \, ds = \int_0^{l(\gamma)} \varrho \circ \gamma_s(t) \, dt.$$

Definition 2.1. Let X be a connected metric space and $G \subsetneq X$ be a non-empty open set. For any $x \in G$, we denote by $\delta_G(x)$ the distance between x and the boundary of G. That is,

$$\delta_G(x) = \operatorname{dist}(x, \partial G).$$

Remark 2.2. In Definition 2.1, the boundary of G is not empty. Otherwise, G is both open and closed which contradicts that X is connected. Hence

$$\partial G \neq \emptyset.$$

For $0 < r < \delta(x)$, the ball $U(x, r)$ is not necessarily contained in G. Thus, we need to consider G as a metric space whose metric is the restriction of the metric of X. The closed and open balls in G with the center x and the radius r are denoted by:

$$\begin{aligned} B_G(x, r) &= \{y \in G : |y - x| \le r\} \\ &= B(x, r) \cap G; \\ U_G(x, r) &= \{y \in G : |y - x| < r\} \\ &= U(x, r) \cap G. \end{aligned}$$

Definition 2.3. Let γ be a rectifiable curve in an open set $G \subsetneq X$. The *quasihyperbolic length* of γ in G is

$$l_{qh}(\gamma) = \int_\gamma \frac{ds}{\delta_G(x)}.$$

The *quasihyperbolic distance* between x and y in G is defined by

$$k_G(x,y) = \inf_\gamma l_{qh}(\gamma),$$

where γ runs over all rectifiable curves in G joining x and y. If there is no rectifiable curve in G joining x and y, we define

$$k_G(x,y) = +\infty.$$

Definition 2.4. Let X be a metric space. An open set D of X is said to be *rectifiably connected* if, for any two points $x, y \in D$, there is a rectifiable curve in D joining x and y.

If $G \subsetneq X$ is a rectifiably connected open set, it is clear that $k_G(x,y) < \infty$ for any two points $x, y \in G$. Thus it is easy to verify that $k_G(\cdot)$ is a metric in G, called the *quasihyperbolic metric* of G.

Definition 2.5. For $c \geq 1$, a metric space X is *c-quasiconvex* if each pair of points $x, y \in X$ can be joined by an curve γ with length $l(\gamma) \leq c|x-y|$.

Observation 2.6. If X is a c-quasiconvex metric space, then any domain $G \subsetneq X$ is is rectifiably connected.

Proof. Choose $x_0 \in G$ and define

$$D_{x_0} = \big\{ y \in G : \text{ there is a rectifiable curve in } G \text{ joining } x_0 \text{ and } y \ \big\}.$$

It is clear that $x_0 \in D_{x_0}$.

We claim that D_{x_0} and $G \backslash D_{x_0}$ are both open in G. For any $y_0 \in D_{x_0}$, since G is open, there exists a $r > 0$ such that $B(y_0, r) \subseteq G$. If $z \in B\big(y_0, r/c\big)$, by the definition of quasiconvex, then there is a rectifiably curve γ joining y_0 and z with

$$l(\gamma) \leq c|y_0 - z| \leq r.$$

This implies that $\gamma \subseteq G$. Thus,

$$B\big(y_0, r/c\big) \subseteq D_{x_0},$$

which implies D_{x_0} is open in G.

With a similar argument, we can deduce that $G \backslash D_{x_0}$ is also open in G. Since G is connected, we have $D_{x_0} = G$. Therefore, G is rectifiably connected. $\square$

Hereafter we will use the abbreviations $\delta(x) = \delta_G(x)$ and $k(\cdot) = k_G(\cdot)$. The following result gives a basic fact about the function $\delta(x)$ which is necessary for our proofs.

Theorem 2.7. *Let X be a c-quasiconvex metric space and let $G \subsetneq X$ be a domain. Then*

(1) $|x-y| \leq \left(e^{k(x,y)} - 1\right) \delta(x), \quad \forall x, y \in G;$

(2) *If $z \in G$, $0 < t \leq 1/2$, and $x, y \in B_G\big(z, t\delta(z)/(4c)\big)$, then*

$$\frac{1}{1+2t} \frac{|x-y|}{\delta(z)} \leq k(x,y) \leq \frac{c}{1-t} \frac{|x-y|}{\delta(z)}. \tag{2.1}$$

Proof.

(1) By Observation 2.6, we know that G is rectifiably connected. For any rectifiable curve γ joining x and y in G, let $\gamma_s : [0, L] \to G$ be the arclength parametrization of γ with $\gamma_s(0) = x$. We have, for each $t \in [0, L]$,

$$\delta\left(\gamma_s(t)\right) \le \delta(x) + |\gamma_s(t) - x| \le \delta(x) + l\left(\gamma_s(t)|_{[0,t]}\right) = \delta(x) + t.$$

Hence

$$l_{qh}(\gamma) \ge \int_0^L \frac{dt}{\delta(x) + t} \ge \ln\left(1 + \frac{|x-y|}{\delta(x)}\right).$$

So we obtain (1).

(2) Suppose that $x, y \in B_G\big(z, t\delta(z)/(4c)\big)$. Since X is a c-quasiconvex space, there is a rectifiable curve γ in X joining x to y with $l(\gamma) \le c|x - y|$.

For any $u \in \gamma$, it is clear that

$$\begin{aligned} |u - z| &\le |u - x| + |x - z| \\ &\le l(\gamma) + t\delta(z)/(4c) \\ &\le c|x - y| + t\delta(z)/(4c) \\ &< \big((2c + 1)/(4c)\big)t\delta(z) \qquad \big(\text{since } c \ge 1\big) \\ &< t\delta(z). \end{aligned} \tag{2.2}$$

The inequality (2.2) implies that

$$\gamma \subseteq U\big(z, t\delta(z)\big).$$

We claim: $\gamma \subseteq G$.

Suppose that $\gamma \nsubseteq G$. From the connectedness of γ, it follows that there is a point

$$u_0 \in \partial G \cap \gamma.$$

Combing the inequality (2.2), we get $\operatorname{dist}\big(z, \partial G\big) \le t\delta(z)$ which implies $\delta(z) \le t\delta(z)$. This is a contradiction since $0 < t < 1$ and $\delta(z) > 0$. Hence, our claim is proved.

For each $u \in \gamma$, since $\gamma \subseteq G$, the function $\delta(u)$ is well defined. Furthermore, we have

$$\delta(u) \ge \delta(z) - |u - z| \ge (1 - t)\delta(z).$$

Let $L = l(\gamma)$ and let $\gamma_s : [0, L] \to \gamma$ be the arc length parametrization of γ. Hence

$$k(x, y) \le \int_0^L \frac{dr}{\delta(\gamma_s(r))} \le \frac{L}{(1 - t)\delta(z)} \le \frac{c}{1 - t}\frac{|x - y|}{\delta(z)}. \tag{2.3}$$

Now we prove the left inequality of the inequality (2.1). Since G is rectifiably connected, the set of rectifiable curves joining x and y is not empty. We assume that $\gamma : [a, b] \to G$ is any rectifiable curve joining x and y in G.

Case 1. $\gamma \subseteq B\big(z, 2t\delta(z)\big)$.

Thus, for all $u \in \gamma$,

$$\delta(u) \le |u - z| + \delta(z) \le (1 + 2t)\delta(z).$$

Therefore, it follows

$$
\begin{aligned}
l_{qh}(\gamma) &= \int_0^{l(\gamma)} \frac{dr}{\delta(\gamma_s(r))} \\
&\geq \frac{l(\gamma)}{1+2t}\frac{1}{\delta(z)} \\
&\geq \frac{|x-y|}{(1+2t)\delta(z)},
\end{aligned}
\tag{2.4}
$$

where γ_s is the arc length parametrization of γ.

Case 2. $\gamma \not\subseteq B\big(z, 2t\delta(z)\big)$.

From the connectedness of γ, we know that γ has two sub-curves $\gamma_1, \gamma_2 \subseteq B\big(z, 2t\delta(z)\big)$ joining the spheres $S\big(z, t\delta(z)\big)$ and $S\big(z, 2t\delta(z)\big)$. For any $u \in \gamma_i$ $(i = 1, 2)$ we have $\delta(u) \leq (1 + 2t)\delta(z)$. Since $l(\gamma_i) \geq t\delta(z) \geq 2c|x - y|$, we again obtain (2.4).

This proves (2.1). □

Theorem 2.8. *Let X be a c-quasiconvex metric space and $G \subsetneq X$ be a domain. Suppose that $x, y \in G$ and either $|x - y| \leq \delta(x)/(8c)$ or $k(x, y) \leq 1/8$. Then*

$$
\frac{1}{2c}\frac{|x-y|}{\delta(x)} \leq k(x,y) \leq 2c\frac{|x-y|}{\delta(x)}. \tag{2.5}
$$

Proof. If $|x - y| \leq \delta(x)/(8c)$, then (2.5) follows from Theorem 2.7 with $t = 1/2$.

Thus we may assume that $|x - y| > \delta(x)/(8c)$ and $k(x, y) \leq 1/8$. It follows that

$$
k(x,y) \leq c|x-y|/\delta(x).
$$

So we need only to prove the left inequality in (2.5). Let $\tilde{r} = k(x, y) \leq 1/8$. From the definition of $k(x, y)$ it follows that, for any $\epsilon > 0$, there is a rectifiable curve γ joining x and y in G such that

$$
\int_0^{l(\gamma)} \frac{dt}{\delta(\gamma_s(t))} < \tilde{r} + \epsilon. \tag{2.6}
$$

Here γ_s is the arc length parametrization of γ and $l(\gamma)$ is the length of γ.

For each $t \in [0, l(\gamma)]$, we have

$$
\begin{aligned}
\delta(\gamma_s(t)) &\leq \delta(x) + |\gamma_s(t) - \gamma_s(0)| \\
&\leq \delta(x) + l(\gamma_s|[0,t]) \\
&= \delta(x) + t.
\end{aligned}
$$

Substituting the above estimation into (2.6), we get

$$
\begin{aligned}
\tilde{r} + \epsilon &> \int_0^{l(\gamma)} \frac{dt}{\delta(x)+t} \\
&= \ln\left(1 + \frac{l(\gamma)}{\delta(x)}\right) \\
&\geq \ln\left(1 + \frac{|x-y|}{\delta(x)}\right).
\end{aligned}
$$

Let $\epsilon \to 0$, we obtain

$$
\tilde{r} \geq \ln\left(1 + \frac{|x-y|}{\delta(x)}\right).
$$

Therefore, $|x-y| \le (e^{\tilde{r}}-1)\ \delta(x) \le 2\tilde{r}\ \delta(x)$. It follows that

$$k(x,y) = \tilde{r} \ge \frac{1}{2c}\frac{|x-y|}{\delta(x)},$$

which implies the theorem. □

Corollary 2.9. Let X be a *c-quasiconvex* metric space and $G \subsetneq X$ be a domain. Then the quasihyperbolic metric and the metric of X define the same topology in the domain G.

Theorem 2.10. *Let X be a c-quasiconvex metric space and $G \subsetneq X$ be a domain. Let γ be a rectifiable path in G and let $l_k(\gamma)$ be the length of γ in the metric space $\big(G, k(\cdot)\big)$. Then*

(1) $l_{qh}(\gamma)/c \le l_k(\gamma) \le l_{qh}(\gamma)$.

(2) *The metric space $\big(G, k(\cdot)\big)$ is a 2-quasiconvex metric space.*

Proof. (1) Let γ_s is the arc length parametrization of γ and L is the length of γ. Let $0 = t_0 < t_1 < \cdots < t_n = L$ be a partition of $[0, L]$. Then

$$\sum_{j=1}^{n} k\big(\gamma_s(t_{j-1}), \gamma_s(t_j)\big) \le \sum_{j=1}^{n} l_{qh}\big(\gamma_s|_{[t_{j-1},t_j]}\big) \le l_{qh}(\gamma).$$

Hence $l_k(\gamma) \le l_{qh}(\gamma)$

Now we prove the left inequality in (1). By the definition of $l_{qh}(\gamma)$, it is follows that

$$l_{qh}(\gamma) = \int_0^L g(t)dt,$$

where $g(t) = 1/\delta\big(\gamma_s(t)\big)$. Let $0 < \epsilon < 1/2$. Since g is continuous, a simple compact argument of γ shows that there is a partition $0 = t_0 < t_1 < \cdots < t_n = L$ of $[0, L]$ such that, for $x_i = \gamma_s(t_i)$ and $\gamma_i = \gamma_s|_{[t_{i-1},t_i]}$, we have

$$l_{qh}(\gamma) \le \sum_{i=1}^{n} g(t_i)(t_i - t_{i-1}) + \epsilon$$

and $\gamma_i \subseteq U\big(x_i,\, \epsilon\,\delta(x_i)/(8c)\big)$ for all $1 \le i \le n$.

For each $1 \le i \le n$, we choose successive points $x_{i-1} = x_{i,0},\ x_{i,0}, \cdots, x_{i,n_i} = x_i$ of γ_i such that

$$l(\gamma_i) \le \sum_{j=1}^{n_i} |x_{i,j-1} - x_{i,j}| + \epsilon/n.$$

With the aid of the estimate (2.1) in Theorem 2.7, we get that, for all $1 \le i \le n$ and $1 \le j \le n_i$,

$$\frac{|x_{i,j-1} - x_{i,j}|}{\delta(x_i)} \le c(1+2\epsilon)k\big(x_{i,j-1}, x_{i,j}\big).$$

Since $t_i - t_{i-1} = l(\gamma)$ and $g(t_i) = 1/\delta(x_i)$, these estimates imply

$$l_{qh}(\gamma) \le c(1+2\epsilon)\sum_i \sum_j k\big(x_{i,j-1}, x_{i,j}\big) + 2\epsilon.$$

Here the double sum is at most $l_k(\gamma)$. Since ϵ is arbitrary, this yields the desired inequality.

(2) Let $a \neq b$ be two points in G. Choose a path γ joining a and b with $l_{qh}(\gamma) < 2k(a,b)$. By (1), we get $l_k(\gamma) < 2k(a,b)$, which implies that $\big(G, k(\cdot,\cdot)\big)$ is 2-quasiconvex.

$\square$

3. Quasisymmetry and ring property

In order to prove Theorem 1.6, we need the following fact.

Fact 3.1. Let $G \subsetneq X$ be a domain in a c-quasiconvex metric space X. For any $x \in G$, if $0 < r < \delta(x)$, then

$$G \backslash B_G(x,r) \neq \emptyset \quad \text{and} \quad S_G(x,r) \neq \emptyset.$$

Proof. By Remark (2.1), we know that $\partial G \neq \emptyset$ and $\delta(x)$ is well defined. If $G \backslash B_G(x,r) = \emptyset$, then

$$G \subseteq B_G(x,r) \subseteq B(x,r).$$

It follows that $\overline{G} \subseteq B(x,r)$ which implies $\partial G \subseteq B(x,r)$. Thus,

$$\delta(x) = \text{dist}\,\big(x, \partial G\big) \leq r,$$

which is a contradiction. Hence, $G \backslash B_G(x,r) \neq \emptyset$.

Since $\partial G \neq \emptyset$ and $0 < r < \delta(x)$, we can choose a point $z' \in \partial G$ such that

$$|z' - x| > r.$$

From the definition of c-quasiconvex of X, there is a curve $\gamma : [a,b] \to X$ joining the points x and z'. Set

$$W := \{t \in [a,b] : \gamma|_{[a,t]} \subseteq U_G(x,r)\}.$$

Since $U_G(x,r) = U(x,r) \cap G$ is an open set of X, it is clear that W is an open set of $[a,b]$ and $a \in W$. Define

$$t_0 = \sup\{t : t \in W\}.$$

Clearly, $t_0 > a$. Since $\gamma|_{[a,t_0)} \subseteq U_G(x,r) \subseteq G$, we know that

$$\gamma(t_0) \in \overline{G} \quad \text{and} \quad |\gamma(t_0) - x| \leq r. \tag{3.1}$$

We claim that $\gamma(t_0) \in S_G(x,r)$.

Since $\gamma(t_0) \in \overline{G}$, we know that

$$\gamma(t_0) \in G \cup \partial G.$$

If $\gamma(t_0) \in \partial G$, by (3.1), then we get

$$\text{dist}(x, \partial G) = \delta(x) \leq r,$$

which contradicts $r < \delta(x)$. Thus, we have

$$\gamma(t_0) \in G \quad \text{and} \quad |\gamma(t_0) - x| \leq r.$$

If $|\gamma(t_0) - x| < r$, by the definition of t_0, then $t_0 \in W$. This is a contradiction since W is an open set. Therefore we have

$$\gamma(t_0) \in G \quad \text{and} \quad |\gamma(t_0) - x| = r,$$

which implies that $\gamma(t_0) \in S_G(x,r)$. Hence, $S_G(x,r) \neq \emptyset$. $\square$

Let X, Y be c, c'-quasiconvex metric spaces and let $G \subsetneq X, G' \subsetneq Y$ be two domains. If $x \in G$ and if $M > 0$, $\alpha > 1$ be two positive real numbers, we denote

$$r_{x,\alpha} = \frac{\delta(x)}{\alpha}, \quad B^G_{x,r} = B_G(x,r) \quad \text{and} \quad \alpha U^G_{x,r} = U_G(x, \alpha r).$$

Definition 3.2. We say that a homeomorphism $f : G \to G'$ has (M, α)-ring property if

$$\sup_{0<r<r_{x,\alpha}} \left\{ \frac{\text{diameter}\big(f(B^G_{x,r})\big)}{\text{dist}\big(f(B^G_{x,r}),\ G' \backslash f(\alpha U^G_{x,r})\big)} \right\} \le M.$$

Remark 3.3. Since $\alpha r < \delta(x)$ and Fact 3.1, we know that $G' \backslash f(\alpha U^G_{x,r}) \neq \emptyset$ which implies Definition 3.2 is well defined.

Proof of Theorem 1.6. In what follows, we will divide the proof into four steps, and prove the result step by step.

Step 1. *We prove that f has $(2H^2(H+1), 3)$-ring property.*

Suppose that $x \in G$ and $0 < r < r_{x,3}$. By Fact 3.1, we know that

$$S_G(x,r) \neq \emptyset \quad \text{and} \quad G \backslash \big(3U^G_{x,r}\big) \neq \emptyset.$$

Suppose that a, b are any two points in $B^G_{x,r}$. Let y be any point on $S_G(x,r)$ and let z be any point in $G \backslash \big(3U^G_{x,r}\big)$.

Claim 1.1. $\text{diameter}\big(f(B^G_{x,r})\big) \le 2H^2 |f(z) - f(y)|$.

Since f is H-quasisymmetric, it follows from the definition that

$$\frac{\sup\{|f(u) - f(x)| \,:\, u \in B_G(x,r)\}}{\inf\{|f(u) - f(x)| \,:\, u \in G \backslash U_G(x,r)\}} \le H.$$

Meanwhile, since $a \in B_G(x,r)$ and $y \in S_G(x,r)$, we have

$$\begin{aligned} |f(a) - f(x)| &\le \sup\{|f(u) - f(x)| \,:\, u \in B_G(x,r)\} \\ &\le H \cdot \inf\{|f(u) - f(x)| \,:\, u \in G \backslash U_G(x,r)\} \\ &\le H|f(y) - f(x)|. \end{aligned}$$

With a similar argument, we obtain

$$|f(b) - f(x)| \le H|f(y) - f(x)|.$$

Therefore,

$$|f(a) - f(b)| \le 2H|f(y) - f(x)|. \tag{3.2}$$

Moreover, since $|z - y| \ge |x - y|$, by using the definition of H-quasisymmetry, we have

$$|f(x) - f(y)| \le H|f(z) - f(y)|. \tag{3.3}$$

From (3.2) and (3.3), we deduce $|f(a) - f(b)| \le 2H^2|f(z) - f(y)|$. Hence, Claim 1.1 is proved.

Let c be any point in B_r.

Claim 1.2. $|f(z) - f(y)| \le (H+1)|f(z) - f(c)|$.

Sice $c \in B^G_{x,r}$, it follows that

$$\begin{aligned} &|y - c| \le 2r, \quad |z - c| \ge 2r \quad \text{and} \\ &|f(z) - f(y)| \le |f(z) - f(c)| + |f(c) - f(y)|. \end{aligned} \tag{3.4}$$

Hence, by the definition of quasisymmetry, we have

$$\begin{aligned}|f(y)-f(c)| &\le \sup\{|f(v)-f(c)| \,:\, v\in B_G(c,r)\}\\ &\le H\cdot \inf\{|f(v)-f(c)| \,:\, v\in G\backslash U_G(c,r)\}\\ &\le H|f(c)-f(z)|.\end{aligned} \tag{3.5}$$

So, from (3.4) and (3.5), Claim 1.2 is obtained.

Combing Claim 1.1 with Claim 1.2, we have thus proved that

$$\text{diameter}\big(f(B^G_{x,r})\big) \le 2H^2(H+1)|f(c)-f(z)|.$$

Since c and z are arbitrary, it follows that

$$\text{diameter}\big(f(B^G_{x,r})\big) \le 2H^2(H+1)\cdot \text{dist}\big(f(B^G_{x,r}), G'\backslash f(3U^G_{x,r})\big).$$

Therefore, f has $(2H^2(H+1),3)$-ring property. □

Let X,Y,G,G',f be as in Theorem 1.6. Remark 2.2 implies that

$$\partial G' \neq \emptyset.$$

Let $x\in G$, $r_{x,2c+1}=\delta(x)/(2c+1)$ and $B^G_{x,r}=B_G(x,r)$. In order to prove our result, we need the following lemma.

Lemma 3.4. *If $0<r<r_{x,2c+1}$ and $z'\in\partial G'$, then*

$$dist\big(z', f(B^G_{x,r})\big) > 0.$$

Proof. Otherwise, there is a point $z'_0\in\partial G'$ such that $\text{dist}\big(z'_0, f(B^G_{x,r})\big)=0$.

It follows that there are points $\{y_n\}$ in $f(B^G_{x,r})$ with $y_n\to z_0$. Furthermore, there are points $\{x_n\}$ in $B^G_{x,r}$ such that $f(x_n)=y_n$.

Step 3.2.1. We show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

If it is not so, then there is a positive real number $\epsilon_0>0$ such that, for each $k\in\mathbb{N}$, there is $j(k)>k$ with $|x_k-x_{j(k)}|\ge\epsilon_0$. Let $k\in\mathbb{N}$ and let $z_k\in\{x_k,x_{j(k)}\}$ be a point with $|z_k-x_1|\ge\epsilon_0/2$.

Claim 3.2. For all that $k\in\mathbb{N}$,

$$|f(x_1)-f(x_{j(1)})| \le M|f(z_k)-f(x_1)|.$$

Here the constant M depends only on H,c,r,ϵ_0.

We distinguish two cases to prove Claim 3.2.

Case 1. $|x_1-x_{j(1)}|\le|z_k-x_1|$. It follows from the definition of H-quasisymmetry that $|f(x_1)-f(x_{j(1)})|\le H|f(z_k)-f(x_1)|$.

Case 2. $|x_1-x_{j(1)}|>|z_k-x_1|$. Since the metric space X is c-quasiconvex, we can join x_1 to $x_{j(1)}$ by a curve $\gamma:[a,b]\to X$ with $l(\gamma)\le c|x_1-x_{j(1)}|$. For any point $p\in\gamma$, it is clear that

$$\begin{aligned}|p-x| &\le |p-x_1|+|x_1-x|\\ &\le l(\gamma)+r\\ &\le c|x_1-x_{j(1)}|+r \quad \big(\text{since } x_1,x_{j(1)}\in B^G_{x,r}\big)\\ &\le (2c+1)r.\end{aligned} \tag{3.6}$$

If $\gamma \nsubseteq G$, from the connectedness of γ, then there exists a point $p_0 \in \gamma \cap \partial G$. By the inequality (3.6), we deduce that

$$\delta(x) = \mathrm{dist}(x, \partial G) \leq (2c+1)r,$$

which contradicts the assumption $r < r_{x,2c+1}$. Hence $\gamma \subseteq G$.

Let $\gamma_s : [0, L] \to G$ be the arc length parametrization of γ where L is the length of γ and $L \leq c|x_1 - x_{j(1)}|$. Define inductively the successive points $x_1 = p_0, p_1, \cdots, p_{s-1}, p_s = x_{j(1)}$ of γ_s as follows. Let $t_0 = 0$,

$$t_j = \sup_t \{t \in [0, L] \ : \ \gamma_s(t) \in B_G\big(p_{j-1}, |z_k - x_1|\big)\}$$

and $p_j = \gamma_s(t_j)$, $0 \leq j \leq s$.

From the construction of t_j, it is clear that $t_i - t_{i-1} \geq |z_k - x_1|$ for $1 \leq i \leq s-1$. Thus, we have $L \geq t_{s-1} \geq (s-1)|z_k - x_1|$. In addition,

$$\begin{aligned} s &\leq \frac{L}{|z_k - x_1|} + 1 \qquad (\text{since } |z_k - x_1| \geq \epsilon_0/2) \\ &\leq \frac{2L}{\epsilon_0} + 1 \\ &\leq \frac{2c|x_1 - x_{j(1)}|}{\epsilon_0} + 1 \qquad (\text{since } x_1, x_{j(1)} \in B(x,r)) \\ &\leq \frac{4cr}{\epsilon_0} + 1. \end{aligned} \tag{3.7}$$

Moreover, since $|x_1 - x_{j(1)}| > |z_k - x_1|$, we have $s \geq 2$, $|p_{j-1} - p_j| = |z_k - x_1|$ for $1 \leq j \leq s-1$ and $|p_{s-1} - p_s| \leq |z_k - x_1|$. Since f is H-quasisymmetric in G and $p_0 = x_1$, we have

$$\begin{aligned} &|f(p_1) - f(p_0)| \leq H|f(z_k) - f(x_1)|; \\ &|f(p_2) - f(p_1)| \leq H|f(p_1) - f(p_0)| \leq H^2|f(z_k) - f(x_1)|; \\ &\qquad \cdots \\ &|f(p_s) - f(p_{s-1})| \leq H|f(p_{s-1}) - f(p_{s-2})| \leq \cdots \leq H^s|f(z_k) - f(x_1)|. \end{aligned}$$

Summation gives

$$|f(x_{j(1)}) - f(x_1)| \leq (H + H^2 + \cdots + H^s)|f(z_k) - f(x_1)| \leq sH^s|f(z_k) - f(x_1)|.$$

Combining this estimate with (3.7), we now have

$$|f(x_1) - f(x_{j(1)})| \leq M_1|f(z_k) - f(x_1)|, \tag{3.8}$$

where M_1 depends only on H, c, r, ϵ_0.

Since $z_k \in \{x_k, x_{j(k)}\}$, by repeated use of the above argument, we can deduce

$$|f(x_1) - f(z_k)| \leq M_2|f(x_k) - f(x_{j(k)})|, \tag{3.9}$$

where M_2 depends only on H, c, r, ϵ_0. Therefore, using (3.8) and (3.9), we now obtain

$$|f(x_1) - f(x_{j(1)})| \leq M|f(x_k) - f(x_{j(k)})|,$$

where M depends only on H, c, r, ϵ_0.

Let $k \to \infty$, we get $|f(x_k) - f(x_{j(k)})| \to 0$ as $f(x_k) = y_k \to z_0$ and $j(k) > k$. Thus, we obtain $f(x_1) = f(x_{j(1)})$. It follows that $x_1 = x_{j(1)}$ which contradicts $|x_{j(1)} - x_1| \geq \epsilon_0 > 0$.

Therefore, the sequence of $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in B_r.

Step 3.2.2. We prove $z'_0 \in G'$.

Since X is complete, there is a point $x_0 \in X$ such that $x_n \to x_0$. As $\{x_n\} \subset B^G_{x,r}$, it follows that $x_0 \in \overline{G} \cap B(x,r)$.

If $x_0 \in \partial G$, then

$$\begin{aligned}\operatorname{dist}\big(x, \partial G\big) &\le |x - x_0| \\ &\le r,\end{aligned}$$

which contradicts the assumption $r < \delta(x)$. Hence $x_0 \in G$.

Since f is continuous, it follows that $f(x_n) = y_n \to f(x_0) \in G'$. Thus $z'_0 = f(x_0) \in G'$ as $y_n \to z'_0$. Hence $z'_0 \in G'$ which contradicts that $z'_0 \in \partial G'$ and G' is an open set.This completes the proof of Lemma 3.3. □

Let us proceed with the proof of Theorem 1.6.

Step 2. *We show that f satisfies*

$$\frac{|f(x) - f(y)|}{\delta'(f(x))} \le \theta\left(\frac{|x-y|}{\delta(x)}\right), \tag{3.10}$$

for all $x, y \in G$ with

$$|x - y| < \frac{\delta(x)}{3^3(2c+1)}.$$

Here

$$\theta(t) = \frac{24(\ln 3)H^5(H+1)^2 c'}{\ln\big(1/(2c+1)t\big)}.$$

Suppose that $|x - y| = t \cdot \delta(x)$, where $0 < t < 1/\big(3^3(2c+1)\big)$. Let m be the largest integer with

$$3^m(2c+1)\,t < 1. \tag{3.11}$$

Then $m \ge 3$. Set

$$r_j = 3^j\, t\, \delta(x),\ U^G_j = U_G(x, r_j) \quad \text{and} \quad B^G_j = B_G(x, r_j) \quad \text{for} \quad 0 \le j \le m. \tag{3.12}$$

Choose a point $z' \in \partial G'$ with

$$|z' - f(x)| \le 2\,\delta'\big(f(x)\big). \tag{3.13}$$

Since Y is a c'-quasiconvex metric space, there is a rectifiable curve $\gamma : [a,b] \to Y$ joining $f(x)$ and z' with

$$\operatorname{length}(\gamma) \le c'|z' - f(x)|. \tag{3.14}$$

Define

$$t_j = \sup_t \big\{t \in [a,b] : \gamma|_{[a,t]} \subseteq f(B^G_j)\big\} \quad \text{and} \quad y_j = \gamma(t_j). \tag{3.15}$$

Since U^G_j is an open set of X and f is a homomorphism from G onto G', $f(U^G_j)$ is an open set of Y. As $\gamma(a) = f(x) \in f(U^G_j)$, it follows that $t_j > a$. Meanwhile, by (3.11) and (3.12), we have

$$r_j < r_{x,2c+1}.$$

which implies $\gamma(b) = z' \in \partial G'$. Together with Lemma 3.4, it follows that $\operatorname{dist}\big(\gamma(b), f(B_j)\big) > 0$. Therefore $t_j < b$.

We prove $a < t_j < b$.

Claim 2.1. $y_j = \gamma(t_j) \in f\big(S_G(x, r_j)\big)$.

We divide the proof of Claim 2.1 into two steps.

Step 2.1. y_j is a boundary point of $f\big(B_j^G\big)$ in Y, that is, $y_j \in \partial_Y f\big(B_j^G\big)$.

Suppose $N \subseteq Y$ is an open neighborhood of y_j. Then $\gamma^{-1}(N)$ is an open neighborhood of t_j. Since $a < t_j < b$, there exists a positive number $\sigma > 0$ such that

$$(t_j - \sigma, t_j + \sigma) \subset \gamma^{-1}(N).$$

From the definition of t_j, it is clear that $\gamma|_{[a,t_j)} \subseteq f(B_j^G)$. Thus we have

$$\begin{aligned}&\gamma\big((t_j - \sigma, t_j)\big) \cap f(B_j^G) \neq \emptyset;\\ &\gamma\big([t_j, t_j + \sigma)\big) \cap \big(Y \setminus f(B_j^G)\big) \neq \emptyset.\end{aligned}$$

Therefore $N \cap f(B_j^G) \neq \emptyset$ and $N \cap (Y \setminus f(B_j^G)) \neq \emptyset$. Then $y_j \in \partial_Y\big(f(B_j^G)\big)$ follows. Note that $\partial_Y\big(f(B_j^G)\big)$ is the boundary of $f(B_j^G)$ in Y(not in G').

Step 2.2. We show that $y_j \in f\big(S_G(x, r_j)\big)$.

Since $y_j \in \partial_Y f(B_j^G)$, it follows that

$$\operatorname{dist}\big(y_j, f(B_j^G)\big) = 0. \tag{3.16}$$

As $f(B_j^G) \subseteq G'$, we know that y_j belongs to the closure of G'.

If $y_j \in \partial_Y G'$, by Lemma 3.4, we get $\operatorname{dist}(y_j, f(B_j^G)) > 0$, which contradicts (3.16). Hence $y_j \notin \partial_Y G'$. Since $y_j \in \overline{G'}$ and G' is open, it follows that $y_j \in G'$.

Since $f(B_j^G)$ is a relative close set in G', we deduce

$$\begin{aligned}f(B_j^G) =&\text{the closur of } f(B_j^G) \text{ in } \ G'\\ =&\overline{f(B_j^G)} \cap G'\\ =&\big\{f\big(B_j^G\big) \cup \partial_Y f(B_j^G)\big\} \cap G'.\end{aligned} \tag{3.17}$$

By using $y_j \in \partial_Y f(B_j^G)$ and $y_j \in G'$, we get

$$y_j \in f\big(B_j^G\big).$$

Thus, there exists a point $x_j \in B_j^G$ such that $f(x_j) = y_j$.

If $x_j \in U_j^G$, that is $|x_j - x| < r_j$ and $x_j \in G$, then it follows immediately that $y_j = f(x_j)$ is an inner point of $f(B_j^G)$. This fact contradicts that $y_j \in \partial_Y f(B_j^G)$. Hence $x_j \in S_G(x, r_j)$ and Claim 2.1 is proved.

Now suppose $x_j \in S_G(x, r_j)$ such that $f(x_j) = y_j$. Denote

$$\lambda = |y_1 - f(x)|.$$

Claim 2.2. $\lambda \leq 4H^3(H+1)c' \cdot \delta'\big(f(x)\big)/(m-1)$.

Since $|x_1 - x| = r_1$ and $|x_{j-1} - x| \geq r_1$, $j \geq 2$, from the definition of H-quasisymmetry, we have

$$\begin{aligned}|f(x_1) - f(x)| &\leq \sup\big\{|f(v) - f(x)| \,:\, v \in B_G(x, r_1)\big\}\\ &\leq H \cdot \inf\{|f(v) - f(x)| \,:\, v \in G \backslash U_G(x, r_1)\}\\ &\leq H|f(x_{j-1}) - f(x)|.\end{aligned} \tag{3.18}$$

That is

$$\lambda \leq H|y_{j-1} - f(x)| \quad \text{for} \quad j \geq 2.$$

From the Step 1 of the proof of Theorem 1.6, we know that f has $\big(2H^2(H+1),3\big)$-ring property. In view of the fact $3\,r_j<\delta(x)$ and $3\,U^G_{j-1}=U^G_j$, we have, for $2\le j\le m$,

$$\begin{aligned}|y_{j-1}-f(x)| &\le \text{diameter}\big(f(B^G_{j-1})\big)\\ &\le 2H^2(H+1)\ \text{dist}\big(f(B^G_{j-1}),G'\backslash f(3U^G_{j-1})\big)\\ &= 2H^2(H+1)\ \text{dist}\big(f(B^G_{j-1}),G'\backslash f(U^G_{j})\big)\\ &\le 2H^2(H+1)|y_{j-1}-y_j|.\end{aligned}\tag{3.19}$$

Therefore, for $j\ge 2$,

$$\lambda\le 2H^3(H+1)|y_{j-1}-y_j|\le 2H^3(H+1)\cdot\text{length}\left(\gamma|_{[t_{j-1},t_j]}\right).$$

Summing over these j and noting (3.13) and (3.14), we obtain

$$\begin{aligned}(m-1)\lambda &\le 2H^3(H+1)\sum_{j=2}^{m}\text{length}\left(\gamma|_{[t_{j-1},t_j]}\right)\\ &\le 2H^3(H+1)\cdot\text{length}(\gamma)\\ &\le 2H^3(H+1)c'|z'-f(x)|\\ &\le 4H^3(H+1)c'\cdot\delta'\big(f(x)\big).\end{aligned}\tag{3.20}$$

Thus the proof of Claim 2.2 is completed.

Since $y\in G$, $|x-y|=t\cdot\delta(x)$ and $r_0=t\,\delta(x)$, we have $y\in B_0$. Note that f has $(2H^2(H+1),3)$-ring property. We deduce that

$$\begin{aligned}|f(y)-f(x)| &\le \text{diameter}\big(f(B^G_0)\big)\\ &\le 2H^2(H+1)\ \text{dist}\big(f(B^G_0),G'\backslash f(U^G_1)\big)\\ &\le 2H^2(H+1)|f(x)-y_1|=2H^2(H+1)\lambda.\end{aligned}\tag{3.21}$$

Together with (3.20) and (3.21), it follows that

$$\frac{|f(y)-f(x)|}{\delta'\big(f(x)\big)}\le\frac{8H^5(H+1)^2c'}{m-1}.\tag{3.22}$$

Since $3^{m+1}(2c+1)t\ge 1$, we have

$$m+1\ge\frac{\ln\big(1/(2c+1)t\big)}{\ln 3}.$$

Furthermore, $3^3(2c+1)t<1$ implies $3\ln 3<\ln\big(1/(2c+1)t\big)$. Hence

$$m-1\ge\frac{\ln\big(1/(2c+1)t\big)-2\ln 3}{\ln 3}\ge\frac{\ln\big(1/(2c+1)t\big)}{3\ln 3}.\tag{3.23}$$

By combing (3.22) with (3.23), we have the desired estimate

$$\frac{|f(y)-f(x)|}{\delta'\big(f(x)\big)}\le\frac{24\ln 3H^5(H+1)^2c'}{\ln\big(1/(2c+1)t\big)}=\theta(t)=\theta\left(\frac{|x-y|}{\delta(x)}\right).\tag{3.24}$$

Here

$$\theta(t)=\frac{24\ln 3\,H^5(H+1)^2c'}{\ln\big(1/(2c+1)t\big)}.$$

It is obvious that $\theta(t)$ is an increasing function and $\theta(t)\to 0$ as $t\to 0$.

Since $\theta(t) \to 0$ as $t \to 0$, we choose a constant $t_1 > 0$ such that

$$t_1 < \frac{1}{3^3 2c(2c+1)} \quad \text{and} \quad \theta(2ct_1) \leq \frac{1}{8c}. \tag{3.25}$$

Define the function $\phi(t)$ as follows:

$$\phi(t) = 2c\,\theta(2ct) = \frac{48\,c\,H^5(H+1)^2 c' \ln 3}{\ln \dfrac{1}{2c(2c+1)t}}. \tag{3.26}$$

Step 3. *We prove that*

$$k'\big(f(x), f(y)\big) \leq \phi\big(k(x,y)\big) \tag{3.27}$$

for all $x, y \in G$ *with*

$$k(x,y) \leq t_1.$$

From the definition of t_1, it follows $t_1 < 1/8$. Suppose that $x, y \in G$ with $k(x,y) \leq t_1$. Thus, by Theorem 2.8, we have

$$\frac{|x-y|}{\delta(x)} \leq 2ck(x,y) \leq 2ct_1 < \frac{1}{3^3(2c+1)}. \tag{3.28}$$

From the conclusion of Step 2 and (3.25), (3.28), it follows that

$$\begin{aligned} \frac{|f(x)-f(y)|}{\delta'(f(x))} &\leq \theta\left(\frac{|x-y|}{\delta(x)}\right) \\ &\leq \theta(2ct_1) \leq 1/(8c). \end{aligned}$$

Applying Theorem 2.8 again, we get that

$$\begin{aligned} k'\big(f(x), f(y)\big) &\leq \frac{2c|f(x)-f(y)|}{\delta'(f(x))} \\ &\leq 2c\theta\left(\frac{|x-y|}{\delta(x)}\right) \\ &\leq 2c\theta\big(2c \cdot k(x,y)\big) \\ &= \phi\big(k(x,y)\big). \end{aligned}$$

This proves Step 3.

Define the function $\psi(t)$ as follows:

$$\psi(t) = \begin{cases} \phi(t) + \frac{2t}{t_1}[\phi(t_1) - \phi(t_1/2)] & \text{for} \quad 0 < t \leq t_1/2; \\ 2A(t - t_1/2) + B & \text{for} \quad t_1/2 \leq t \leq t_1; \\ At + B & \text{for} \quad t_1 \leq t, \end{cases} \tag{3.29}$$

where

$$A = \frac{2\phi(t_1)}{t_1} \quad \text{and} \quad B = \phi(t_1). \tag{3.30}$$

Then the function ψ has all the properties which are stated in Theorem 1.6.

Step 4. *We show that*

$$k'\big(f(x), f(y)\big) \leq \psi\big(k(x,y)\big) \tag{3.31}$$

for all $x, y \in G$.

Since G is rectifiable connected, by Theorem 2.8, we know that $\big(G, k(\cdot)\big)$ is 2-quasiconvex. Therefore, for any given $x, y \in G$, there is a path $\gamma \subset G$ joining x and y with $l_k(\gamma) \leq 2k(x,y)$.

Let $p \geq 0$ be the unique integer satisfying

$$p\, t_1 < l_k(\gamma) \leq (p+1)\, t_1.$$

Let $\gamma_s^k : [0, l_k(\gamma)] \to G$ be the arc length parametrization of γ with metric $k(\cdot)$. Denote

$$t_j = \frac{j}{p+1} l_k(\gamma) \quad \text{and} \quad x_j = \gamma_s^k(t_j)$$

for all $0 \leq j \leq p+1$. Thus,

$$k\big(x_{j-1}, x_j\big) \leq l_k\big(\gamma_s^k|_{[t_{j-1}, t_j]}\big) = t_j - t_{j-1} = \frac{l_k(\gamma)}{p+1} \leq t_1.$$

By using the conclusion of Step 3, it follows that

$$k'\big(f(x_{j-1}), f(x_j)\big) \leq \phi\big(k(x_{j-1}, x_j)\big) \leq \phi(t_1),$$

for all $1 \leq j \leq p+1$. Hence

$$k'\big(f(x), f(y)\big) \leq \sum_{j=1}^{p+1} k'\big(f(x_{j-1}), f(x_j)\big) \leq (p+1)\phi(t_1). \tag{3.32}$$

Since $p\, t_1 < l_k(\gamma) \leq 2k(x,y)$, it follows from (3.32) that, for all $x, y \in G$,

$$k'\big(f(x), f(y)\big) \leq A\, k(x,y) + B. \tag{3.33}$$

If $0 < k(x,y) \leq t_1/2$, using the conclusion of Step 3, we know that

$$k'\big(f(x), f(y)\big) \leq \phi\big(k(x,y)\big) \leq \psi\big(k(x,y)\big).$$

If $t_1/2 < k(x,y) \leq t_1$, by the conclusion of Step 3, we deduce that

$$k'\big(f(x), f(y)\big) \leq \phi\big(k(x,y)\big) \leq \phi(t_1) = B \leq \psi\big(k(x,y)\big).$$

If $t_1 < k(x,y)$, according to (3.33), we get that

$$k'\big(f(x), f(y)\big) \leq Ak(x,y) + B = \psi\big(k(x,y)\big).$$

Therefore, we obtain

$$k'\big(f(x), f(y)\big) \leq \psi\big(k(x,y)\big),$$

for all $x, y \in G$. Hence, Theorem 1.6 is proved. □

4. Quasi-invariance of quasihyperbolic metric implies quasiconformality

Proof of Theorem 1.8. Define a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ by

$$\eta(t) = e^t - 1.$$

Suppose that $x, y \in G$, $|y - x| = t \cdot \delta_G(x)$ with $0 < t < 1/(8c)$. It follows immediately from Theorem 2.7 and Theorem 2.8 that

$$k(y,x) \leq 2c\, t, \ \frac{|f(y) - f(x)|}{\delta_{G'}\big(f(x)\big)} \leq \eta\big(k'(f(y), f(x))\big).$$

It follows from Theorem 1.8 that

$$\frac{|f(y) - f(x)|}{\delta_{G'}\big(f(x)\big)} \leq \tilde{\theta}\left(\frac{|y - x|}{\delta_G(x)}\right),$$

where $\tilde{\theta}(t) = \eta \circ \varphi(2ct)$. Note that $\tilde{\theta}$ is an increasing function.

Let $a \in G$, $|a - x| \leq \delta_G(x)/(8c)$ and let $b \in G$ with $|a - x| \leq |b - x|$. In what follows, we distinguish two cases to prove

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \tilde{\theta}(1). \tag{4.1}$$

Case 1. $|b - x| > \delta_G(x)$.

Denote $D = G\backslash\{b\}$ and $D' = f(D)$. Since G is a non-cut-point domain, by using Observation 2.6, we know that D is a sub-domain of G. Since $|b - x| > \delta_G(x)$, it follows that $\delta_D(x) = \delta_G(x)$. Thus, we have

$$\frac{|a - x|}{\delta_D(x)} = \frac{|a - x|}{\delta_G(x)} < 1.$$

Applying Theorem 1.8 to the sub-domain D, we get

$$\frac{|f(a) - f(x)|}{\delta_{D'}\big(f(x)\big)} \leq \tilde{\theta}\left(\frac{|a - x|}{\delta_D(x)}\right) \leq \tilde{\theta}(1).$$

Since $\delta_{D'}\big(f(x)\big) \leq |f(b) - f(x)|$, it follows immediately that

$$|f(a) - f(x)| \leq \tilde{\theta}(1)\delta_{D'}\big(f(x)\big) \leq \tilde{\theta}(1)|f(b) - f(x)|.$$

Case 2. $|b - x| \leq \delta_G(x)$.

Denote $D = G\backslash\{b\}$ and $D' = f(D)$. It is clear that $\delta_D(x) = |b - x|$. Thus, it follows that

$$\frac{|a - x|}{\delta_D(x)} = \frac{|a - x|}{|b - x|} \leq 1.$$

Use a similar argument in Case 1, we can obtain the inequality (4.1).

Now, it follows immediately from the inequality (4.1) that

$$\limsup_{r \to 0} H_f(x, r) \leq \tilde{\theta}(1).$$

Hence, the mapping f is a H-quasiconformal mapping with

$$H = \tilde{\theta}(1) = e^{\varphi(2c)} - 1.$$

5. The composition of two quasisymmetric mappings is quasiconformal

Proof of Theorem 1.9. Set

$$\alpha_0 = (2c_1 + 1)3^n, \tag{5.1}$$

where

$$n = 8[c_1 H_1^2(H_1 + 1)] + 9.$$

Claim: The map $g \circ f$ has $\big(2H_2^2(H_2 + 1), \alpha_0\big)$-ring property.

Let $x \in G$ and $\delta_G(x) = \text{dist}(x, \partial G)$. Suppose that

$$r_{x,\alpha_0} = \frac{\delta_G(x)}{\alpha_0} \quad \text{and} \quad 0 < r < r_{x,\alpha_0}.$$

Define

$$r_j = 3^j r, \quad U_j = U_G(x, r_j) \quad \text{and} \quad B_j = B_G(x, r_j)$$

for all $0 \leq j \leq n$. Denote

$$R = \text{diameter}\big(f(B_0)\big).$$

Step 5.1. We show that

$$dist\Big(f(B_0), G'\backslash f(\alpha_0 U_0)\Big) > 3R \quad and \quad 3R < \delta_{G'}\big(f(x)\big). \tag{5.2}$$

From Fact 3.1, it follows that

$$G'\backslash f\big(\alpha_0\, U_0\big) \neq \emptyset.$$

Suppose that $y_0 \in f(B_G)$ and $z \in G'\backslash f\big(\alpha_0\, U_0\big)$ are any two points. Since Y is a c_2-quasiconvex metric space, there exists a rectifiable curve $\gamma : [a,b] \to Y$ jointing y_0 and z with length$(\gamma) \le c_2|y_0 - z|$.

For $1 \le j \le n$, let

$$t_j = \sup_t \Big\{t \in [a,b] \,:\, \gamma|_{[a,t]} \subseteq f(B_j)\Big\} \quad \text{and} \quad y_j = \gamma(t_j). \tag{5.3}$$

Since $(2c_1+1)r_j < \delta_G(x)$, by repeated use of the argument in Step 2 in the proof Theorem 1.6, we deduce that $a < t_j < b$ and

$$y_j = \gamma(t_j) \in f\big(S_G(x, r_j)\big) \quad \text{for all} \quad 1 \le j \le n. \tag{5.4}$$

Since f is homomorphic, it follows that

$$y_i \neq y_j \quad \text{for} \quad i \neq j. \tag{5.5}$$

From the definition of t_j and 5.4, it is clear that

$$\gamma|_{[a,t_j]} \subseteq f(B_j) \subseteq f(U_{j+1}).$$

Thus, we have $t_j < t_{j+1}$. Again, this implies that

$$a < t_1 < t_2 < \cdots < t_n < b. \tag{5.6}$$

In addition, since f has $\big(2H_1^2(H_1+1), 3\big)$-ring property, $y_{j-1} \in f(3^{j-1}B_0)$ and $y_j \in G'\backslash f(3^j U_0)$, we deduce that, for $2 \le j \le n$,

$$\begin{aligned}
\text{length}\big(\gamma|_{[t_{j-1},t_j]}\big) &\ge |y_{j-1} - y_j| \\
&\ge \text{dist}\Big(f(3^{j-1}B_0), G'\backslash f(3^j U_0)\Big) \\
&\ge \frac{1}{2H_1^2(H_1+1)}\text{diameter}\big(f(3^{j-1}B_0)\big) \\
&\ge \frac{1}{2H_1^2(H_1+1)}\text{diameter}\big(f(B_0)\big).
\end{aligned} \tag{5.7}$$

Summing these inequalities from $j = 2$ to n, by noting (5.6), we obtain

$$\text{length}(\gamma) \ge \frac{n-1}{2H_1^2(H_1+1)}\text{diameter}\big(f(B_0)\big).$$

From the definition (5.1) of n and length$(\gamma) \le c_2|y_0 - z|$, it follows that

$$\begin{aligned}
|y_0 - z| &\ge \frac{1}{c_2}\text{length}(\gamma) \\
&\ge \frac{n-1}{2c_2H_1^2(H_1+1)}\text{diameter}\big(f(B_0)\big) \\
&\ge 4\text{diameter}\big(f(B_0)\big).
\end{aligned} \tag{5.8}$$

Since y_0 and z are arbitrary, we obtain

$$\text{dist}\Big(f(B_0), G'\backslash f(3^n U_0)\Big) > 3 \text{ diameter}\big(f(B_0)\big). \tag{5.9}$$

Thus, the first inequality of Step 5.1 is obtained.

By repeated use of the above argument, we deduce that

$$3R < \operatorname{dist}\Big(f(x), \partial G'\Big),$$

that is,

$$3R < \delta_{G'}\big(f(x)\big).$$

Step 5.2. We claim that

$$B_{G'}\big(f(x), 3R\big) \subseteq f\big(\alpha_0 U_0\big).$$

Suppose that

$$B_{G'}\big(f(x), 3R\big) \nsubseteq f\big(\alpha_0 U_0\big).$$

Then there exists a point $\tilde{y}$ with $\tilde{y} \in B_{G'}\big(f(x), 3R\big)$ and $\tilde{y} \in G' \backslash f\big(\alpha_0 U_0\big)$. From (5.9), it follows that

$$\begin{aligned} |f(x) - \tilde{y}| &\geq \operatorname{dist}\big(f(B_0), G' \backslash f(\alpha_0 U_0)\big) \\ &> 3R, \end{aligned}$$

which contradicts $\tilde{y} \in B_{G'}\big(f(x), 3R\big)$. This proves Step 5.2.

Denote

$$B^* = B_{G'}\big(f(x), R\big) \quad \text{and} \quad U^* = U_{G'}\big(f(x), R\big).$$

Step 5.2 implies that

$$3U^* \subseteq f\big(\alpha_0 U_0\big). \tag{5.10}$$

From the definition of R, it is clear that

$$f(B_0) \subseteq B^*. \tag{5.11}$$

Since $g : G' \to G''$ has $\Big(2H_2^2(H_2 + 1), 3\Big)$-ring property, we deduce that

$$\operatorname{diameter}\big(g(B^*)\big) \leq 2H_2^2(H_2 + 1) \cdot \operatorname{dist}\Big(g(B^*), G'' \backslash g\big(3U^*\big)\Big). \tag{5.12}$$

Combing (5.10), (5.11) and (5.12), it follows that

$$\begin{aligned} \operatorname{diameter}\big(g \circ f(B_0)\big) &\leq \operatorname{diameter}\big(g(B^*)\big) \\ &\leq 2H_2^2(H_2 + 1) \cdot \operatorname{dist}\Big(g(B^*), G'' \backslash g\big(3U^*\big)\Big) \\ &\leq 2H_2^2(H_2 + 1) \cdot \operatorname{dist}\Big(g \circ f(B_0), G'' \backslash g \circ f\big(3^n U_0\big)\Big). \end{aligned} \tag{5.13}$$

Therefore, the map $g \circ f$ has $\big(2H_2^2(H_2 + 1), n\big)$-ring property.

Combination of Step 1 to Step 4 in the proof of Theorem 1.6 and Theorem 1.8 gives that $g \circ f$ is a quasiconformal mapping. Hence, Theorem 1.9 is proved.

6. Appendix

For the sake of completeness, we give an example to show that the assumption of non-cut-point in Theorem 1.8 is necessary.

Example 6.1. For each positive integer $n \geq 1$, we define the functions $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} nx & \text{for } x \in [0, \frac{1}{2}] \\ \\ x + \frac{n-1}{2} & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Let $X = \mathbb{R} = Y$ and $G = (0, \infty) = G'$. Define a homeomorphism $f : G \to G'$ as follows:

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in (0,1] \\ f_2(x-1) + f_1(1) & \text{for } x \in [1,2] \\ \cdots\cdots & \\ f_n\big(x-(n-1)\big) + f_{n-1}(n-1) & \text{for } x \in [n-1,n] \text{ and } n \geq 3 \\ \cdots\cdots & \end{cases}$$

By Definition 1.3, it is clear that

$$\limsup_{r \to 0} H_f(n,r) = n+1$$

which implies that $f(x)$ is not a quasiconformal mapping.

In the follows we show that the homeomorphism $f(x)$ satisfies the requirements of Theorem 1.8. Let $x \in G = (0, \infty)$. We write

$$L^*(x,f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y-x|}.$$

Suppose that $x \in (m-1, m]$ for some positive integer m. If $m = 1$, then

$$\begin{aligned} &L^*(x,f) \leq 2, \\ &\delta_G(x) = x \quad \text{and} \\ &\delta_{G'}\big(f(x)\big) = x. \end{aligned}$$

If $m \geq 2$, then

$$\begin{aligned} &L^*(x,f) \leq m+1, \\ &\delta_G(x) = x \leq m \quad \text{and} \\ &\delta_{G'}\big(f(x)\big) \geq (m-1)m/4. \end{aligned}$$

Hence, for all $x \in (0, \infty)$,

$$\frac{L^*(x,f)\, \delta_G(x)}{\delta_{G'}} \leq 12.$$

By Theorem 4.6 of [18], we know that, $\forall x, y \in G$,

$$k_{G'}\big(f(x), f(y)\big) \leq 12 \cdot k_G(x,y).$$

By Theorem 4.7 of [18], it follows that, for any sub-domain $E \subseteq G$ and $\forall x, y \in E$,

$$k_{E'}\big(f(x), f(y)\big) \leq 576 \cdot k_E(x,y),$$

where $E' = f(E)$.

7. Acknowledgements

We wish to express our sincere gratitude to the anonymous referee for his/her careful reading and very useful suggestions.

References

1. Beurling, A. and Ahlfors, L. V., *The boundary correspondence under quasiconformal mappings.* Acta Math., **96** (1956), 125–142.
2. David, G. and Semmes, S., *Strong Aoo-weights, Sobolev inequalities and quasiconformal mappings.* in *Analysis and Partial Differential Equations*, pp. 101-111. Lecture Notes in Pure and Appl. Math., **122**. Dekker, NewYork, 1990.
3. Gehring, F. W. and Osgood, B. G., *Uniform domains and the quasi-hyperbolic metric.* J. Analyse Math., **36** (1979), 50–74.
4. Gehring, F. W. and Palka, B. P., *Quasiconformally homogeneous domains.* J. Analyse Math., **30** (1976), 172–199.
5. Federer, N., *Geometric Measure Theory*, Grundlehren Math. Wiss., **153**, Springer- Verlag, NewYork, 1969.
6. Hajlasz, P., *Sobolev spaces on metric-measure spaces.* Contemporary Mathematics, **338** (2003), 173–218.
7. Heinonen, J. and Koskela, P., *Definitions of quasiconformality.* Invent. Math., **120** (1995), 61–79.
8. Heinonen, J. and Koskela, P., *Quasiconformal maps in metric spaces with controlled geometry.* Acta Math., **181** (1998), 1–61.
9. Heinonen, J., Koskela, P., Shanmugalingam, N. and Tyson, J. T. , *Sobolev classes of Banach space-valued functions and quasiconformal mappings.* J. Analyse. Math., **85** (2001), 87–139.
10. Heinonen, J. and Semmes, S., *Thirty-three yes or no questions about mappings, measures, and metrics.* Conform. Geom. Dyn., **1** (1997), 1–12.
11. Tukia, P. and Väisälä, J., *Quasisymmetric embeddings of metric spaces.* Ann. Acad. Sci. Fenn. Math., **5** (1980), 97–114.
12. Tyson, J., *Quasiconformality and quasisymmetry in metric measure spaces.* Ann. Acad. Sci. Fenn. Math., **23** (1998), 525–548.
13. Tyson, J., *Metric and Geometric Quasiconformality in Ahlfors Regular Loewner Spaces.* Conformal and Dynamics, **5** (2001), 21–73.
14. Väisälä, J., *Lectures on n-Dimensional Quasiconformal Mappings.*, Lecture Notes in Math., **229**. Springer-Verlag, Berlin-NewYork, 1971.
15. Väisälä, J., *Quasisymmetric embeddings in Euclidean spaces.* Trans. Amer. Math. Soc., **264** (1981), 191–204.
16. Väisälä, J., *Quasimöbiüs maps.* J. Analyse Math., **44** (1984/85), 218–234.
17. Väisälä, J., *Quasiconformal maps of cylindrical domains.* Acta Math., **162** (1989), 201–225.
18. Väisälä, J., *Free quasiconformality in Banach spaces, I.* Ann. Aead. Sci. Fenn. Math., **15** (1990), 355–379.
19. Väisälä, J.,*Free quasiconformality in Banach spaces, II.* Ann. Aead. Sci. Fenn. Math., **16** (1991), 255–310.
20. Väisälä, J.,*Free quasiconformality in Banach spaces, III.* Ann. Aead. Sci. Fenn. Math., **17** (1992), 393–408.
21. Väisälä, J.,*Free quasiconformality in Banach spaces, IV.* Analysis and Topology, (1998), 697–717.
22. Väisälä, J.,*The free quasiworld: freely quasiconformal and related maps in Banach spaces.* Quasiconformal geormetry and dynamics (Lublin 1996), Banach Center Publications, Vol. **48**, Polish Academy of Science, Warsaw, 1999, 55-118.
23. Vuorinen, M., *Conformal Geometry and Quasiregular Mappings.* Lecture Notes in Math., **1319** Springer-Verlag, NewYork, 1988.

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