Mathematical Methods in Computational Electromagnetism

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Maxwell equations: A short history

The electrical field is defined as the force experienced by a unit positive charge in space. By Coulomb Law (1785):

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{q}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}, \quad \text{for point charge } \boldsymbol{q} \text{ at } \boldsymbol{y} \in \mathbb{R}^3.$$

For the charge density ρ, the electric field is

$$\boldsymbol{E}(\boldsymbol{x}) = \int_{\mathbb{R}^3} \rho(\boldsymbol{y})\frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}d\boldsymbol{y} = -\nabla \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}d\boldsymbol{y}.$$

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For the charge density ρ, the electric field is

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By studying this function, Gauss finds Gauss divergence theorem and Gauss Law (1813), the first Maxwell equation,

$$\mathrm{div}\boldsymbol{E} = 4\pi\rho.$$

The magnetic field generated by a steady current I over a curve C is given by the Biot-Savart Law (1820)

$$B(x) = \frac{1}{c}\int_C \frac{Idl \times (x-y)}{|x-y|^3}, \quad c = 2.9 \times 10^{10} \ \text{cm/sec.}$$

If J is the current density, then

$$B(x) = \frac{1}{c}\int_{\mathbb{R}^3} \frac{J(y) \times (x-y)}{|x-y|^3} dy = -\frac{1}{c}\int_{\mathbb{R}^3} J(y) \times \nabla_x \left(\frac{1}{|x-y|}\right) dy,$$

Since $\nabla \times (\psi A) = \nabla\psi \times A + \psi\nabla \times A$,

$$B(x) = \frac{1}{c}\nabla \times \int_{\mathbb{R}^3} \frac{J(y)}{|x-y|} dy.$$

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By the identity $\nabla \times \nabla \times A = \nabla(\mathrm{div} A) - \Delta A$,

$$\nabla \times B(x) = \frac{1}{c}\nabla\left(\mathrm{div}\int_{\mathbb{R}^3} \frac{J(y)}{|x-y|} dy\right) - \frac{1}{c}\Delta \int_{\mathbb{R}^3} \frac{J(y)}{|x-y|} dy.$$

For the steady current, $\mathrm{div} J = 0$, this implies by Gauss theorem

$$\nabla \times B(x) = \frac{4\pi}{c} J(x).$$

It is obvious to obtain the fourth Maxwell equation

$$\mathrm{div}\ B = 0.$$

By Faraday Law (1831), for any surface S with boundary C and unit normal vector n

$$\int_C E \cdot dl = -\frac{1}{c}\frac{\partial}{\partial t}\int_S B \cdot n ds(x).$$

By Stokes theorem (1854)

$$\int_C E \cdot dl = \int_S \nabla \times E \cdot n ds(x),$$

one obtains the third Maxwell equation

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0.$$

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[Kelvin, 2 July 1850] letter to Stokes, problem in the exam of Smith Prize in 1854.

$$\mathrm{div} E = 4\pi\rho, \quad \nabla \times B = \frac{4\pi}{c} J, \quad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \quad \mathrm{div} B = 0.$$

The equations are not symmetric in time. The problem is due to the steady current condition $\mathrm{div} J = 0$ in Biot-Savart law.

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Maxwell (1861) replaces J in the second equation by $J + \frac{1}{4\pi}\frac{\partial E}{\partial t}$ to obtain the correct second Maxwell equation

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}J.$$

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***Jinchang Shao, Stokes Theorem and Electomagnetism, Math. Propag, 18 (1994), 6-17.**

Contents

The finite element method: a short histrory

The word "FEM" first appeared in [Clough, 1960], earlier names "direct stiffness method" (Boeing) or "variational difference method" (former USSR and China).

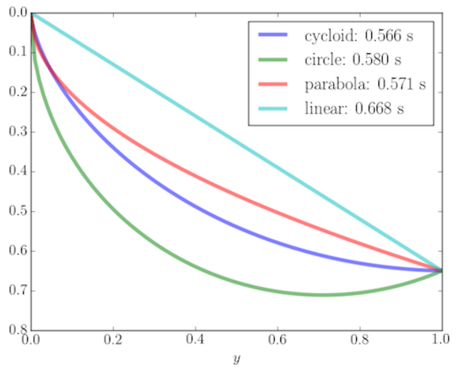
1. Model: The extension of standard structure analysis method in which structure is treated as an assemblage of structure element, no need to consider convergence;

2. Mathematics: An approximation method for solving differential or integral equations, need to consider convergence, error estimation, adaptivity.

Mathematical development of FEM

The approximation method to derive Euler equation for variational problems.

[Leibnitz, 1696] (Letter to Johann Bernoulli) for brachistochrone:



$$T = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx.$$

[Euler, 1807] for the function minimizing the general functional

$$J(y) = \int_a^b Z(x, y, y') dx.$$

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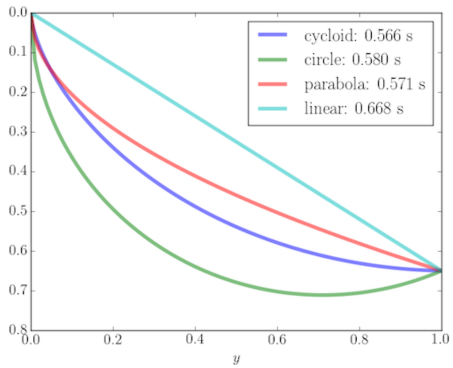

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Variational method for Dirichlet problem of Laplace equation:

$$\min_{u=g \text{ on } \partial\Omega} \int_\Omega |\nabla u|^2 dx.$$

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[Courant, 1943] used the piecewise linear functions, the Courant element, no convergence proof.

Engineering development of FEM

The analysis of frames and structure by computer started the era of modern finite element method in engineering community: [Langefors, 1952], [Turner, 1956], [Argyris, 1960]. NASA started the project of NASTRAN programm in 1965.

[Melosh, 1962] in his PhD thesis recognized the relation of FEM in the sense of [Clough, 1960] and the variational principle.

After 1963, the finite element method as the approximation method dominated. The convergence is taken as for granted.

The finite element method

The variational problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega a(x)\nabla u \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega).$$

The finite element method

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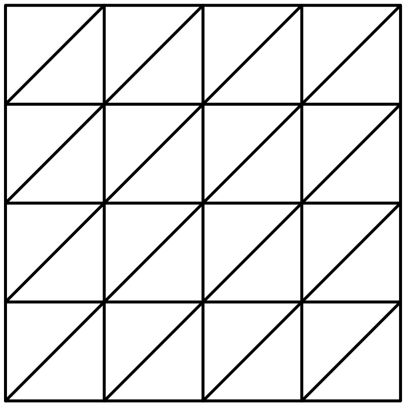
$\mathcal{M}_h$: shape regular mesh of Ω.

$V_h \subset H_0^1(\Omega)$: conforming linear finite element space.

The finite element method: Find $u_h \in V_h$ such that

$$\int_\Omega a(x)\nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v_h \in V_h.$$



A priori error estimate

The error in energy norm satisfies

$$\|\!|u - u_h|\!\|_\Omega \le Ch^\sigma \|u\|_{H^{1+\sigma}(\Omega)}, \quad 0 < \sigma \le 1,$$

where $h = \max_{K \in \mathcal{M}_h} h_K$, and $\|\!|\phi|\!\|_\Omega^2 = \int_\Omega a(x)|\nabla\phi|^2\,dx$.

[Friedrichs, 1962] proved the convergence in H^1 for linear element.
[Oganesjan, 1963] proved the error estimate for H^2 solutions.
[Feng, 1965] convergence, also quadrilateral meshes with hanging nodes.

[Strang-Fix, 1972], [Ciarlet, 1978] text books on finite element method.

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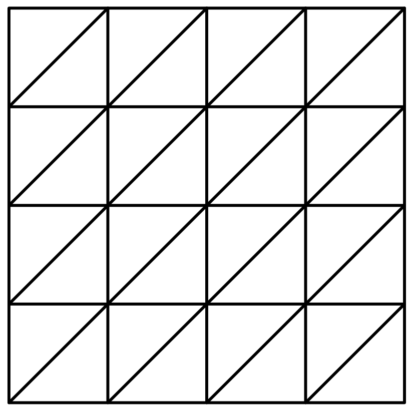
***I. Babuska, Courant element: before and after, In Finite Element Methods: Fifty years of the Courant element, by Michel Krizek, Pekka Neittaanmaki, Rolf Stenberg (Editors), CRC Press, 1994.**

A Posteriori error estimate

Theorem [Babuska-Rheinboldt,1978] Let $a(x) = 1$ in Ω. We have

$$C_1\Big(\sum_{i=1}^{J} \eta_i^2\Big)^{1/2} \le \|u - u_h\|_\Omega \le C_2\Big(\sum_{i=1}^{J} \eta_i^2\Big)^{1/2},$$

where $\{x_i\}_{i=1}^J$ is the set of interior nodes, $\{\psi_i\}_{i=1}^J$ is the set of nodal basis functions of V_h, $S_i = \text{supp}(\psi_i), i = 1, \cdots, J$, and



$$\begin{aligned} -\Delta w_i &= f \text{ in } S_i, \quad w_i = u_h \text{ on } \partial S_i. \\ \eta_i &= \|\nabla(w_i - u_h)\|_{L^2(S_i)}, \end{aligned}$$

Outline of the proof

Notice that $\sum_{i=1}^{J}\psi_i = 1$**, denote** $v = u - u_h \in H_0^1(\Omega)$**, for any** $v_h \in V_h$**,**

$$
\begin{aligned}
\int_\Omega \nabla(u-u_h)\cdot\nabla v dx &= \int_\Omega \nabla(u-u_h)\cdot\nabla(v-v_h)dx \\
&= \sum_{i=1}^{J}\int_{S_i}\nabla(u-u_h)\cdot\nabla[\psi_i(v-v_h)]dx \\
&= \sum_{i=1}^{J}\int_{S_i}\nabla(w_i-u_h)\cdot\nabla[\psi_i(v-v_h)]dx \\
&\le \left(\sum_{i=1}^{J}\eta_i^2\right)^{1/2}\left(\sum_{i=1}^{J}\|\nabla[\psi_i(v-v_h)]\|_{L^2(S_i)}^2\right)^{1/2}.
\end{aligned}
$$

The key is the existence of an interpolation function $v_h \in V_h$ such that

$$\sum_{i=1}^{J} \|\nabla[\psi_i(v - v_h)]\|^2_{L^2(S_i)} \le C\|\nabla v\|^2_{L^2(\Omega)}.$$

The paper uses a complicated construction. A simpler one is known [Clement, 1975]. This is the upper bound. For the lower bound, we have

$$\begin{aligned}
\sum_{i=1}^{J} \eta_i^2 &= \sum_{i=1}^{J} \int_{S_i} \nabla(w_i - u_h) \cdot \nabla(w_i - u_h)dx \\
&= \sum_{i=1}^{J} \int_{S_i} \nabla(w_i - u_h) \cdot \nabla(u - u_h)dx \\
&\le C\left(\sum_{i=1}^{J} \eta_i^2\right)^{1/2} \|\nabla(u - u_h)\|_{L^2(\Omega)}.
\end{aligned}$$

A Posteriori error estimate

The a posteriori error estimate in [Babuska-Rheinboldt,1978] is not fully computable.

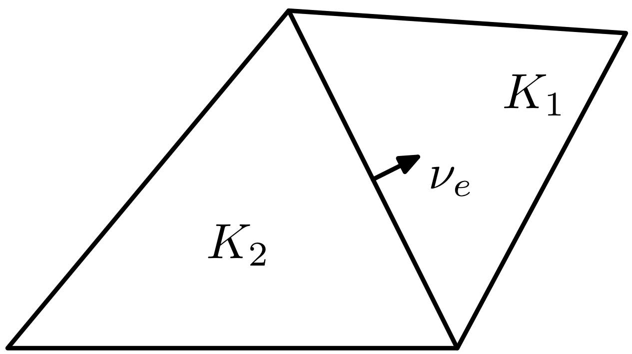
Theorem [Babuska-Miller,1987] We have

$$\|u - u_h\|_\Omega \le C\Big(\sum_{K\in\mathcal{M}_h} \eta_K^2\Big)^{1/2},$$

where

$$\eta_K^2 = \| h_K f \|_{L^2(K)}^2 + \sum_{e\subset\partial K} \| h_e^{1/2} J_e \|_{L^2(e)}^2,$$

$$J_e = \big((a(x)\nabla u_h)|_{K_1} - (a(x)\nabla u_h)|_{K_2}\big)\cdot \nu_e.$$

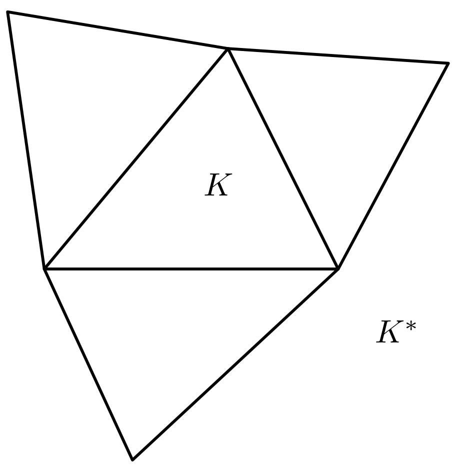


The local lower bound

Theorem [Verfürth,1989] We have

$$\|\!|u - u_h|\!\|_{K^*}^2 \geq C_1 \eta_K^2 - C_2 \sum_{T \subset K^*} \| h_T(f - f_T) \|_{L^2(T)}^2,$$

where $f_T = \frac{1}{|T|} \int_T f \; dx, \quad \forall \; T \in \mathcal{M}_h.$



The adaptive finite element method

$$\|u - u_h\|^2_{H^1(\Omega)} = \sum_{K \in \mathcal{M}_h} \|u - u_h\|^2_{H^1(K)} \approx C \sum_{K \in \mathcal{M}_h} \eta_K^2$$

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Error equi-distribution strategy [Babuska-Rheinboldt,1978]:

$$\text{if} \quad \eta_K > \frac{1}{2} \max_{K \in \mathcal{M}_h} \eta_K, \quad \text{refine } K \in \mathcal{M}_h.$$

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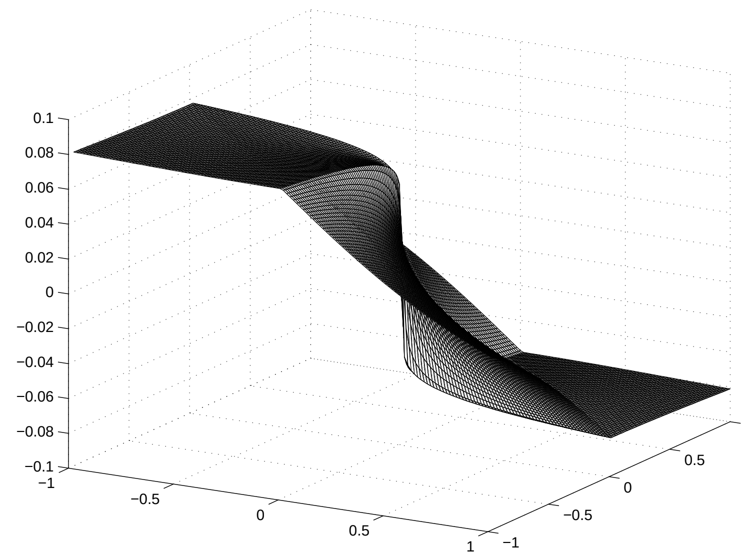
Optimal computation complexity:

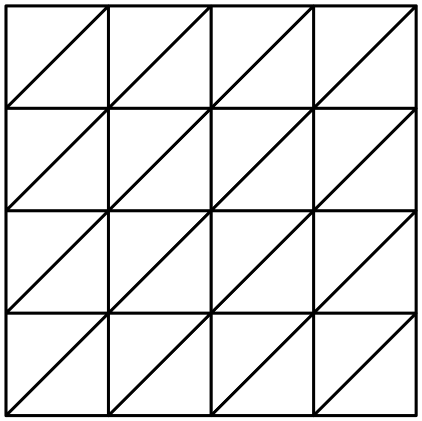
$$\|u - u_h\|_{H^1(\Omega)} = O(N^{-1/d}).$$

Numerical experiments

Let $\Omega = (-1,1) \times (-1,1)$. We solve the equation $-\nabla \cdot (a(x)\nabla u) = 0$ in Ω. Set $a(x) \approx 161.45$ in the first and third quadrants, and $a(x) = 1$ in the second and fourth quadrants.

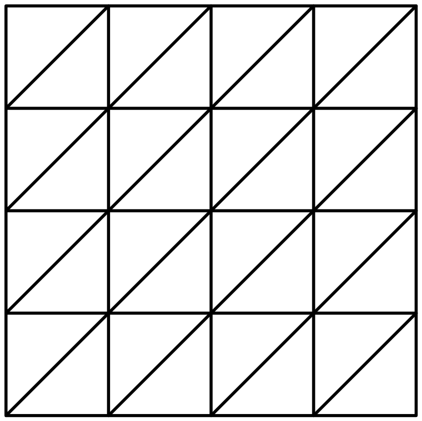
An exact solution is constructed by [Kellogg,1975]: $u = r^{0.1}\mu(\theta)$, μ is smooth. Then $u \in H^{1+\sigma}(\Omega), \sigma < 0.1$.





FEM with uniform mesh

128×128 **mesh:** $|||u - u_h|||_\Omega = 0.8547$



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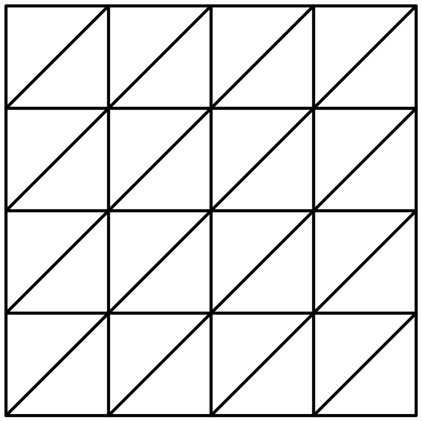
$\mathbf{128 \times 128}$ **mesh:** $\|u - u_h\|_\Omega = \mathbf{0.8547}$

$\mathbf{512 \times 512}$ **mesh:** $\|u - u_h\|_\Omega = \mathbf{0.7981}$



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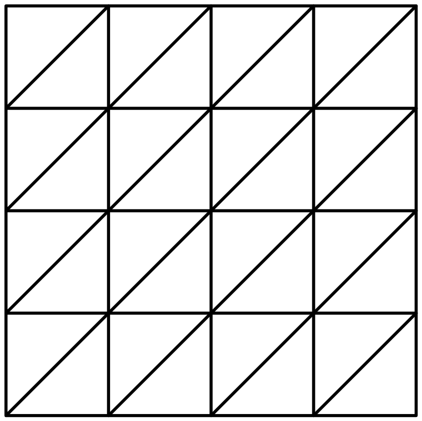
1024×1024 **mesh:** $|||u - u_h|||_\Omega = 0.6954$



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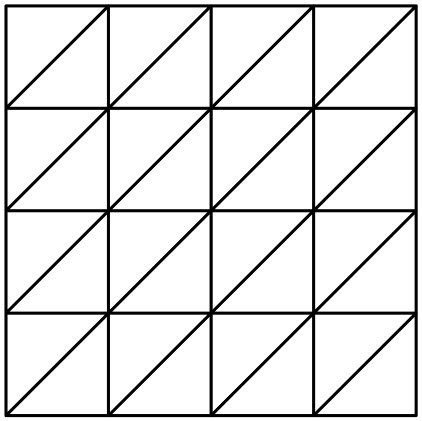
Convergence rate: $h^{0.08}$



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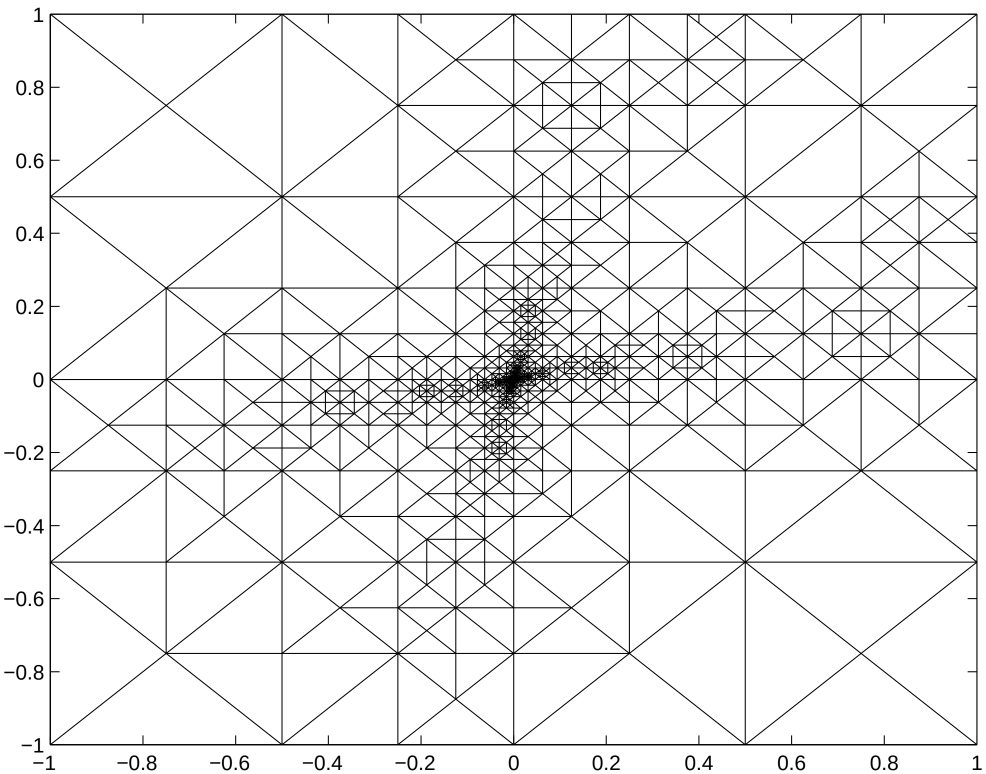
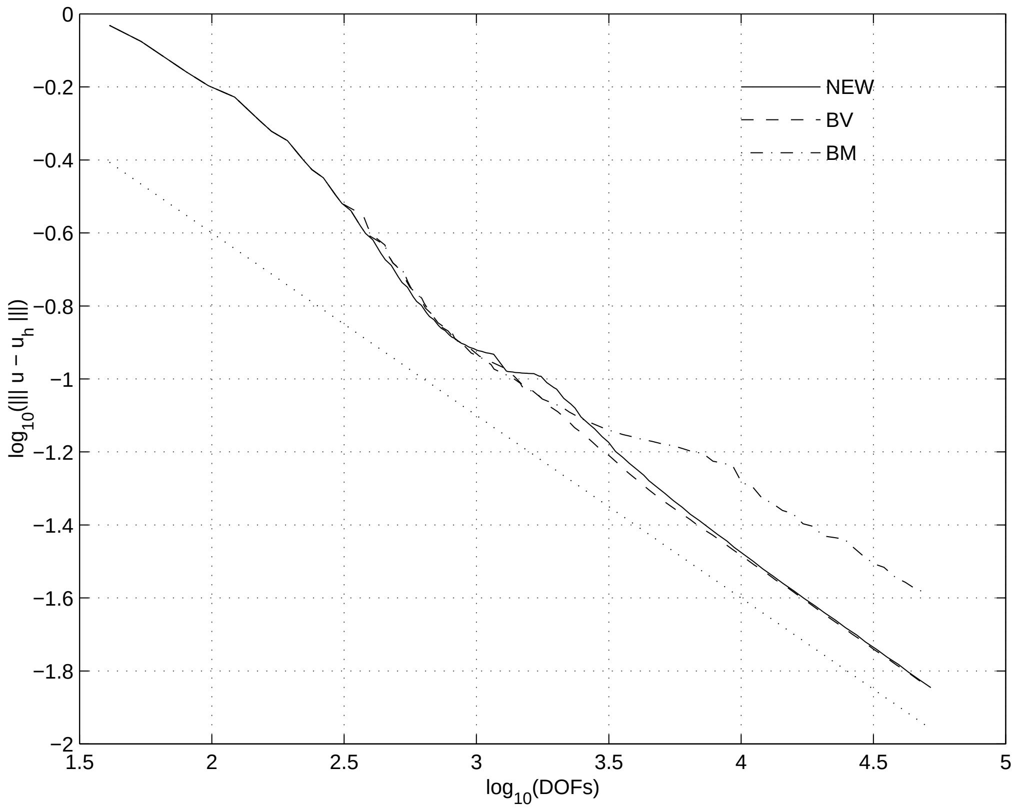
A priori error estimate implies that one must introduce 10^{11} nodes in each space direction to reduce the energy error below 0.1!

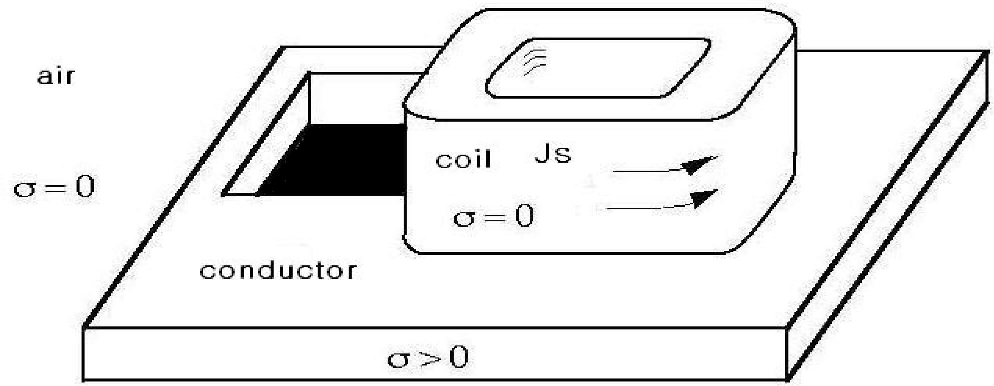


The adaptive mesh of 2673 nodes. The energy error is 0.07451.



Quasi-optimality of the method: $\|\!|u - u_N|\!\|_\Omega \approx CN^{-1/2}$. The quasi-optimal decay is indicated by the dotted line of slope $-1/2$.

The eddy current model: $\mathbf{H}-\psi$ formulation



$$\begin{cases} \textbf{curl}\,\mathbf{H} = \varepsilon\dfrac{\partial \mathbf{E}}{\partial t} + \mathbf{J} & \textbf{in} \quad \mathbb{R}^3, \qquad (\textbf{Ampere-Maxwell's law}) \\ \mu\dfrac{\partial \mathbf{H}}{\partial t} + \textbf{curl}\,\mathbf{E} = 0 & \textbf{in} \quad \mathbb{R}^3, \qquad (\textbf{Farady's law}) \\ \textbf{div}(\mu \mathbf{H}) = 0 & \textbf{in} \quad \mathbb{R}^3, \end{cases}$$

where $\mathbf{J} = \sigma\,\mathbf{E}$ **in** Ω_c, $\mathbf{J}_s$ **in** $\mathbb{R}^3 \setminus \bar{\Omega}_c$.

Literature remarks

Finite element method for Maxwell equations:

- **[Nedelec, 1980]** $H(\mathrm{curl})$ **conforming finite element, edge element**
- **[Hiptmair, 2002] Acta Numerica, [Monk, 2003] Clarendon Press**

A posteriori error estimates for Maxwell equations:

- **[Monk, 1998], [Beck-Deuflhard-Hiptmair-Hoppe-Wohlmuth, 1999] smooth domain**
- **[Zheng-C.-Wang, 2006], [C.-Wang-Zheng, 2007] Birman-Solomyak**
- **[Schöberl, 2007] A new interpolation operator**

A posteriori error estimate

Theorem [Zheng-C.-Wang, 2006] Let $\mathrm{e} = \mathbf{H}(t) - \mathbf{H}_h(t)$.

$$\|\sqrt{\mu}\,\mathbf{e}(t_m)\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{e}\|_{\mathrm{L}^2((0,T);\,\mathrm{L}^2(\Omega))}^2 \le C\sum_{n=1}^{m}\tau_n\left\{\left(\eta_{\mathrm{time}}^n\right)^2 + \left(\eta_{\mathrm{space}}^n\right)^2\right\},$$

where the a posteriori error estimates are given by

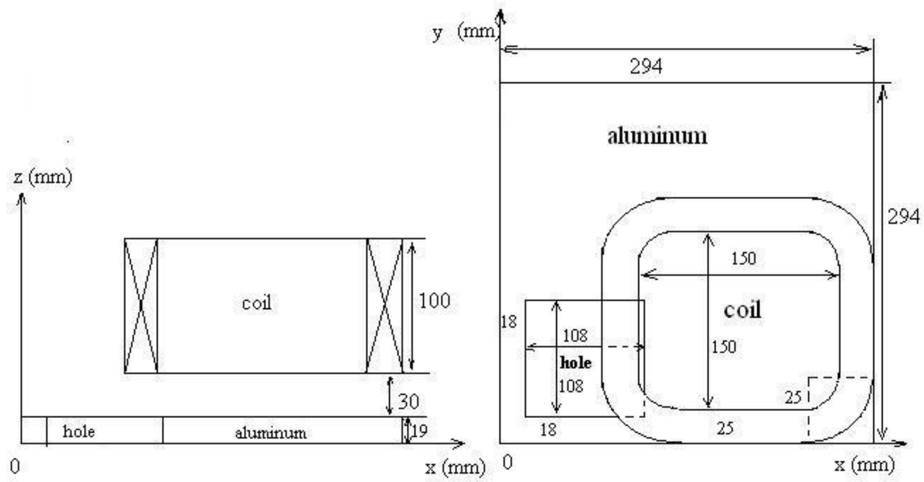
$$\begin{aligned}
\left(\eta_{\mathrm{time}}^n\right)^2 &= \|\mathbf{curl}(\mathbf{H}_n - \mathbf{H}_{n-1})\|_{0,\Omega_c}^2 + \tau_n^{-1}\|\mathbf{f} - \bar{\mathbf{f}}_n\|_{\mathrm{L}^2((t_{n-1},t_n);\,\mathrm{L}^2(\Omega))}^2,\\
\left(\eta_{\mathrm{space}}^n\right)^2 &= \sum_{T\in\mathcal{T}_n}\left(\eta_{0,T}^n\right)^2 + \sum_{T\in\mathcal{T}_n^c}\left(\eta_{1,T}^n\right)^2 + \sum_{F\in\mathcal{F}_n^{\Omega}}\left(\eta_{0,F}^n\right)^2\\
&\quad + \sum_{F\in\mathcal{F}_n^{\Omega c}}\left(\eta_{1,F}^n\right)^2 + \sum_{F\in\mathcal{F}_n^{\partial\Omega}}\left(\eta_{0,B,F}^n\right)^2,
\end{aligned}$$

with the local error indicators defined by

$$
\begin{aligned}
\eta_{0,T}^{n} &:= h_T \left\| div \left(\bar{\mathbf{f}}_n - \mu \frac{\partial \mathbf{H}_h}{\partial t} \right) \right\|_{0,T}, \\
\eta_{1,T}^{n} &:= h_T \left\| \bar{\mathbf{f}}_n - \mu \frac{\partial \mathbf{H}_h}{\partial t} - \mathbf{curl}(\sigma^{-1}\, \mathbf{curl}\, \mathbf{H}_n) \right\|_{0,T}, \\
\eta_{0,F}^{n} &:= \sqrt{h_F} \left\| \left[\left(\bar{\mathbf{f}}_n - \mu \frac{\partial \mathbf{H}_h}{\partial t} \right) \cdot \mathbf{n} \right]_F \right\|_{0,F}, \\
\eta_{1,F}^{n} &:= \sqrt{h_F} \left\| \left[\sigma^{-1} \mathbf{curl}\, \mathbf{H}_n \times \mathbf{n} \right]_{J,F} \right\|_{0,F}, \\
\eta_{0,B,F}^{n} &:= \sqrt{h_F} \left\| \left(\bar{\mathbf{f}}_n - \mu \frac{\partial \mathbf{H}_h}{\partial t} \right) \cdot \mathbf{n} \right\|_{0,F}.
\end{aligned}
$$

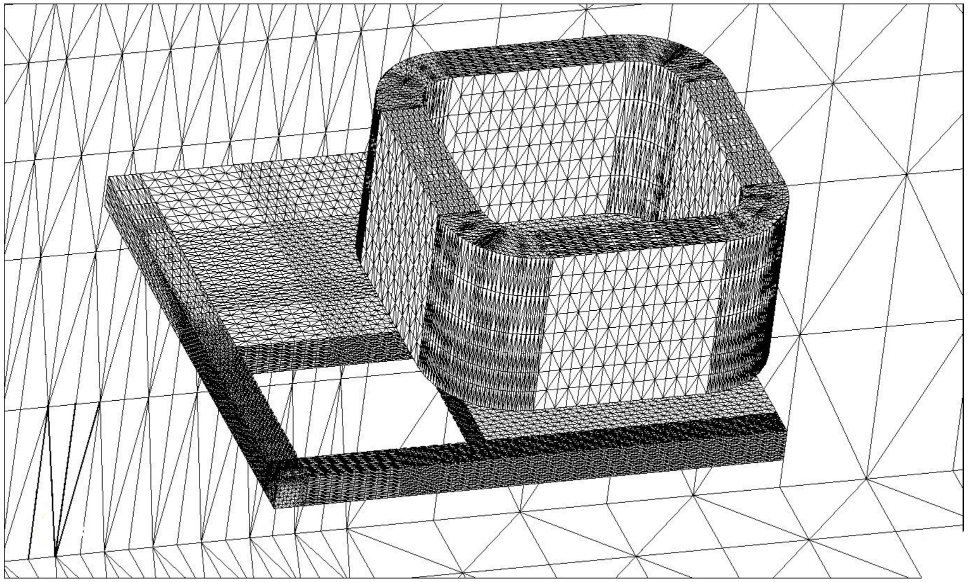
Here $\mathcal{F}_n^{\Omega}$, $\mathcal{F}_n^{\Omega_c}$, and $\mathcal{F}_n^{\partial\Omega}$ denote the edges in Ω, in Ω_c, and on $\partial\Omega$ respectively.

Numerical example

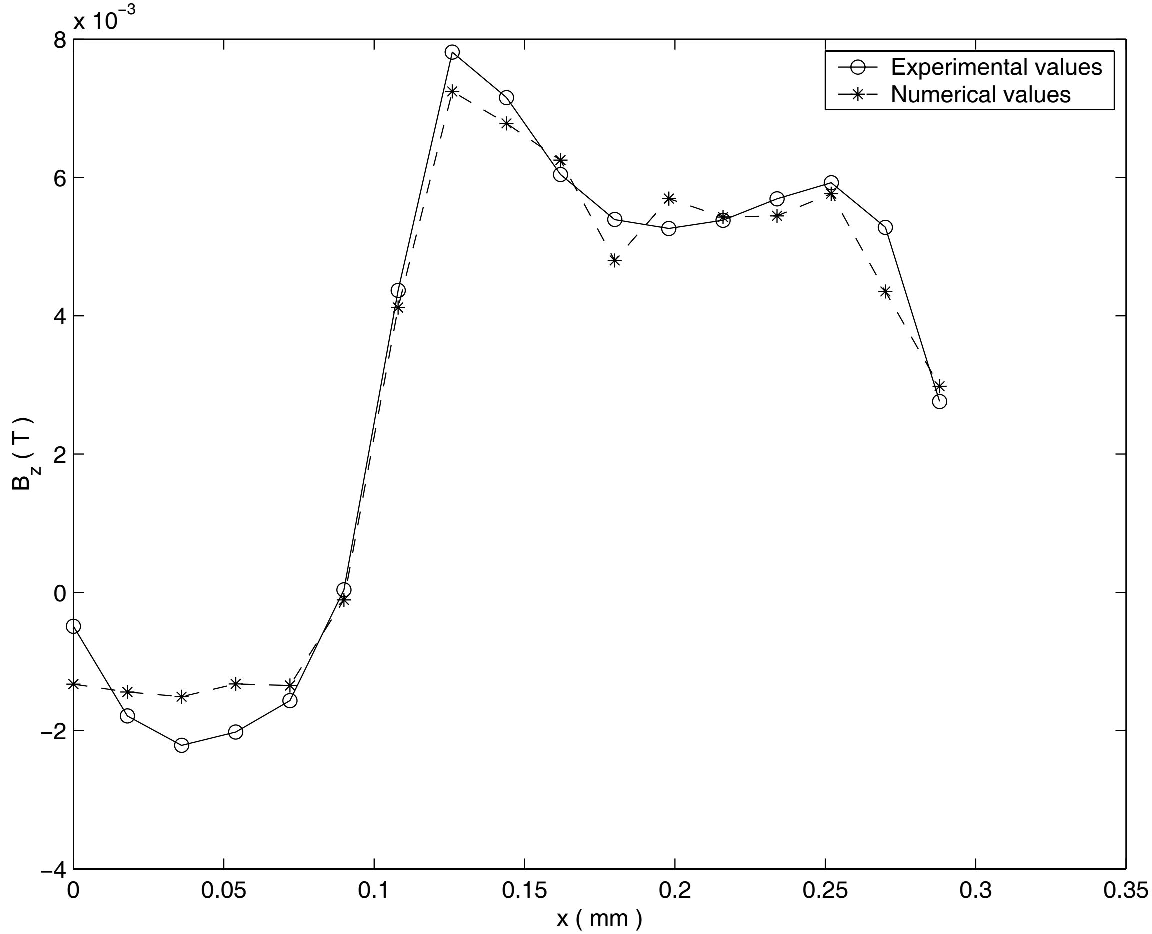


Team Workshop Problem 7. **This problem consists of an aluminum plate with a hole above which a racetrack shaped coil is placed. The aluminum plate has a conductivity of 3.526×10^7 Siemens/Metre and the sinal driving current of the coil is 2742 Ampere/Turn. The frequency of the driving current is $\omega = 50$ Hertz.**

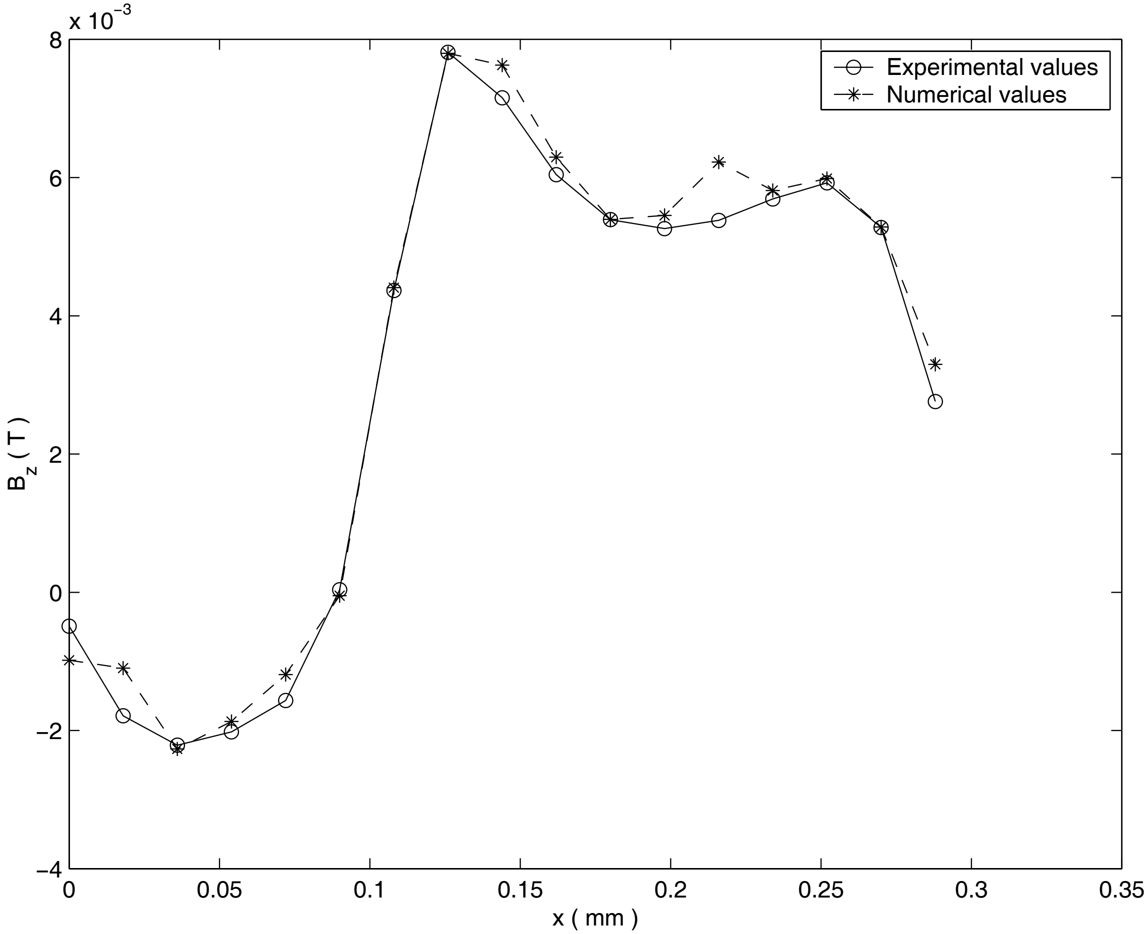
We set Ω to be a cubic domain with one-meter edges and start the computation with zero initial value. We compare the peak values of the vertical magnetic flux μH_z with measured values on some points. These points are located at $y = 72$ mm, $z = 34$ mm, and $x = (18 \times i)$ mm where $i = 0, \cdots, 16$.



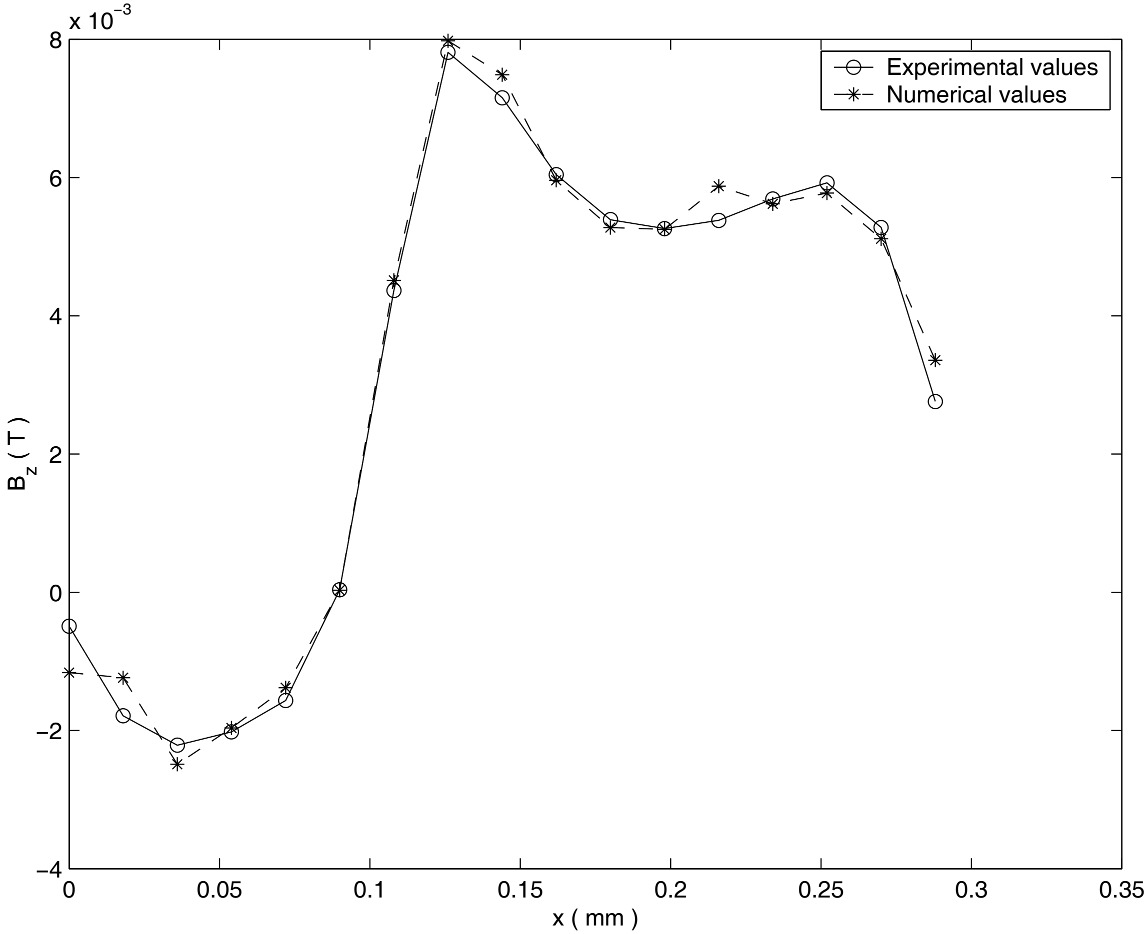
An adaptively refined mesh of 2,263,668 elements after 18 adaptive iterations from 77,760 initial elements.



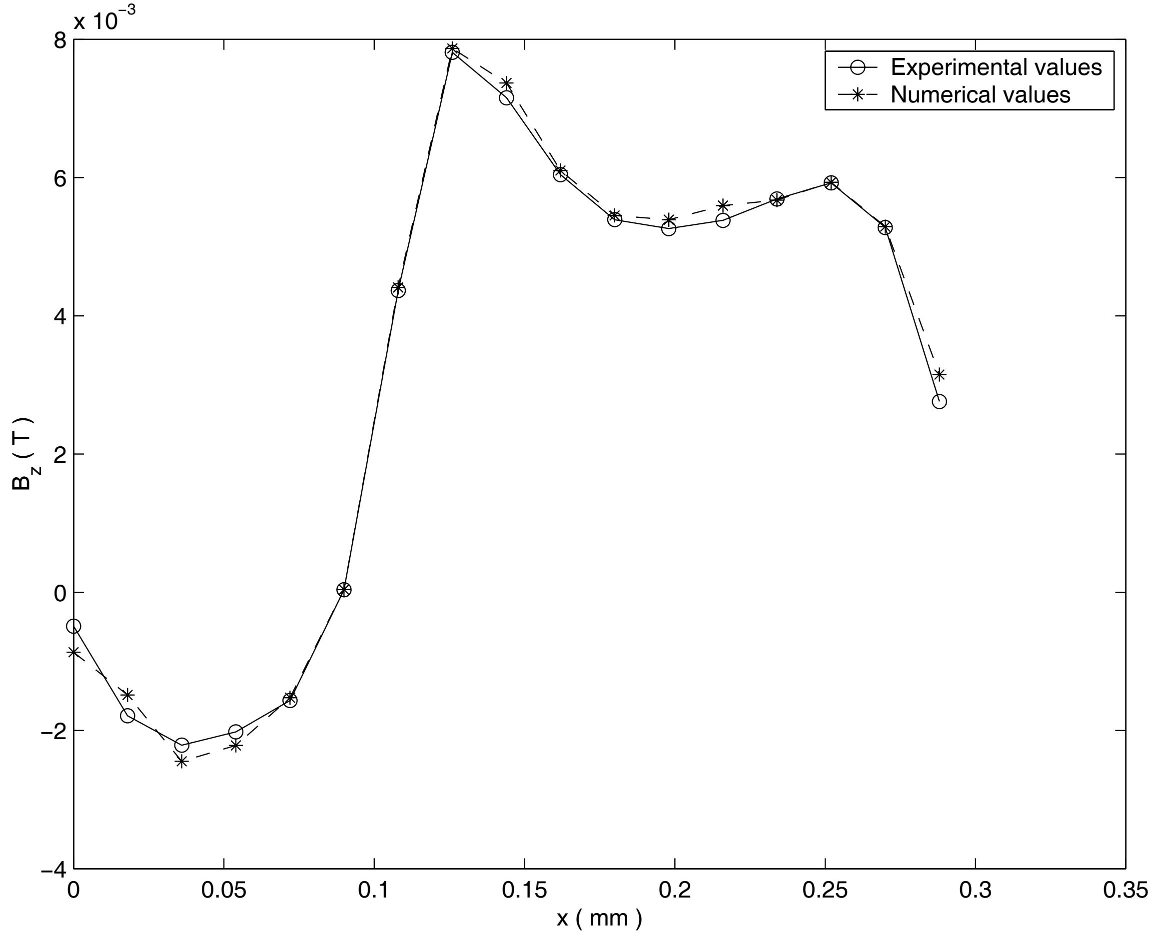
Numerical values of μH_z with $M = 55$, $N_{\text{total}} = 5,555,550$, $\eta_{\text{total}} = 0.0126$, the number of degrees of freedom on $\mathcal{T}_M$ is 36,714.



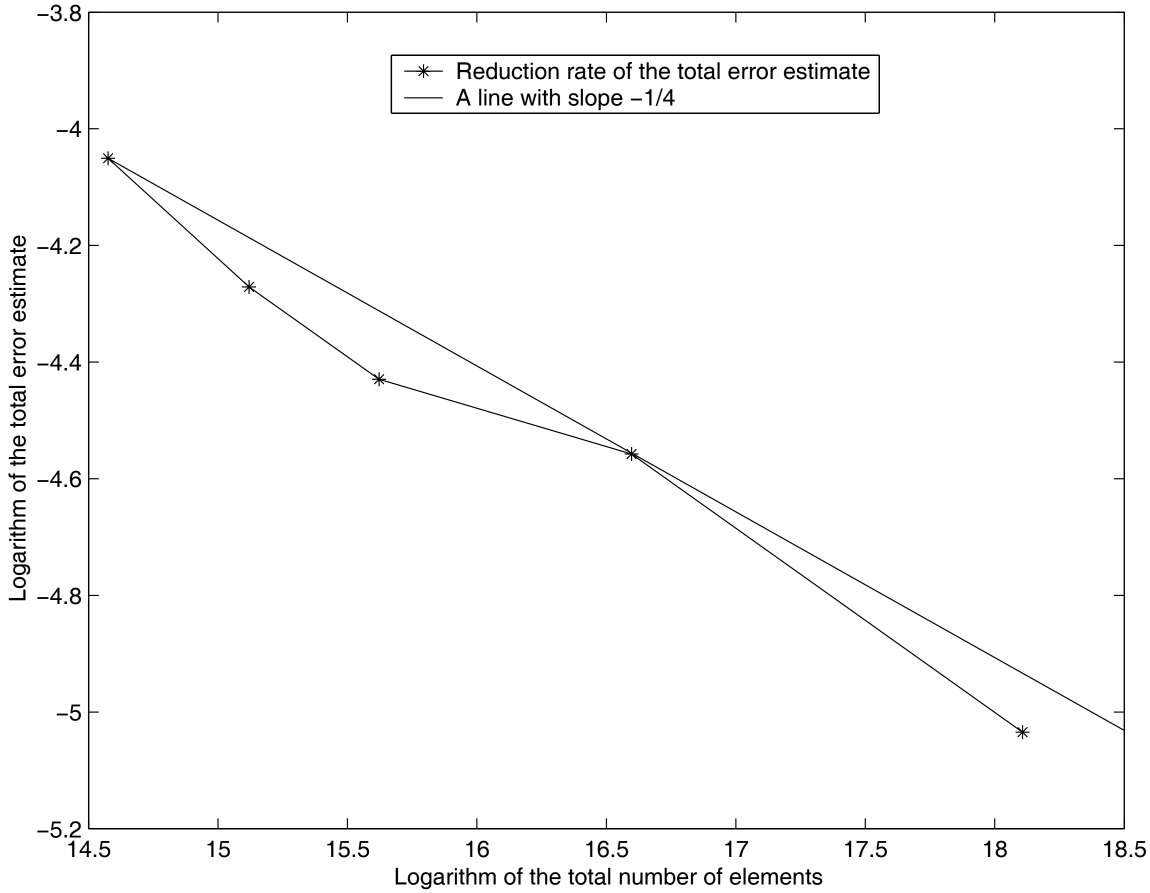
Numerical values of μH_z with $M = 55$, $N_{\text{total}} = 16,152,290$, $\eta_{\text{total}} = 0.0105$, the number of degrees of freedom on $\mathcal{T}_M$ is 120,558.



Numerical values of μH_z with $M = 110$, $N_{\text{total}} = 73,068,160$, $\eta_{\text{total}} = 0.0065$, the number of degrees of freedom on $\mathcal{T}_M$ is 277,883.



Numerical values of μH_z with $M = 110$, $N_{\text{total}} = 249,003,480$, $\eta_{\text{total}} = 0.0047$, the number of degrees of freedom on $\mathcal{T}_M$ is 873,971.



Quasi-optimality of the adaptive mesh refinements of the total a posteriori error estimate.

M. Clemens, J. Lang, D. Teleaga, and G. Wimmer [JCM 2009, 642-656]

Automatic control of discretization errors is quite attractive from a practical point of view. Time consuming validation of numerical solutions usually done through parameter tuning in repeated calculations is no longer needed.

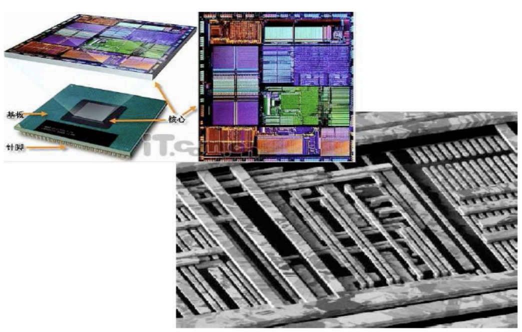
Circuit simulation

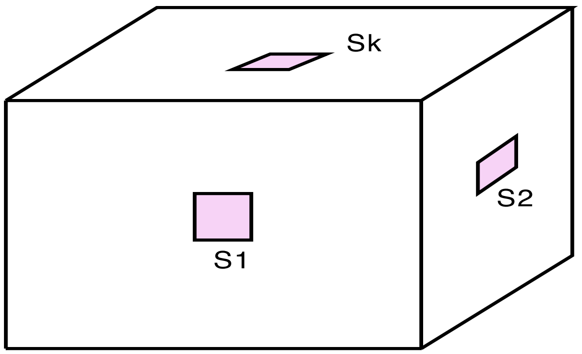
Let $\Omega = \Omega_c \cup \Omega_{nc}$. The Ω_c is fed by N external sinusoidal voltage generators through electrodes $S_1, \cdots, S_N$, $\Gamma = \partial\Omega$ and $\Gamma_e = \cup_{j=1}^N S_j$,

$$\nabla \times \mathbf{E} = -\mathrm{i}\omega\mu\mathbf{H}, \qquad \nabla \times \mathbf{H} = \sigma(x)\mathbf{E},$$

$$(\nabla \times \mathbf{E}) \cdot \nu|_{\Gamma\backslash\Gamma_e} = 0, \qquad \mathbf{E} \times \nu|_{\Gamma_e} = 0.$$

Motivation: Parasitic parameter (resistance, inductance) extraction of large scale integrated circuits





A posteriori error estimate

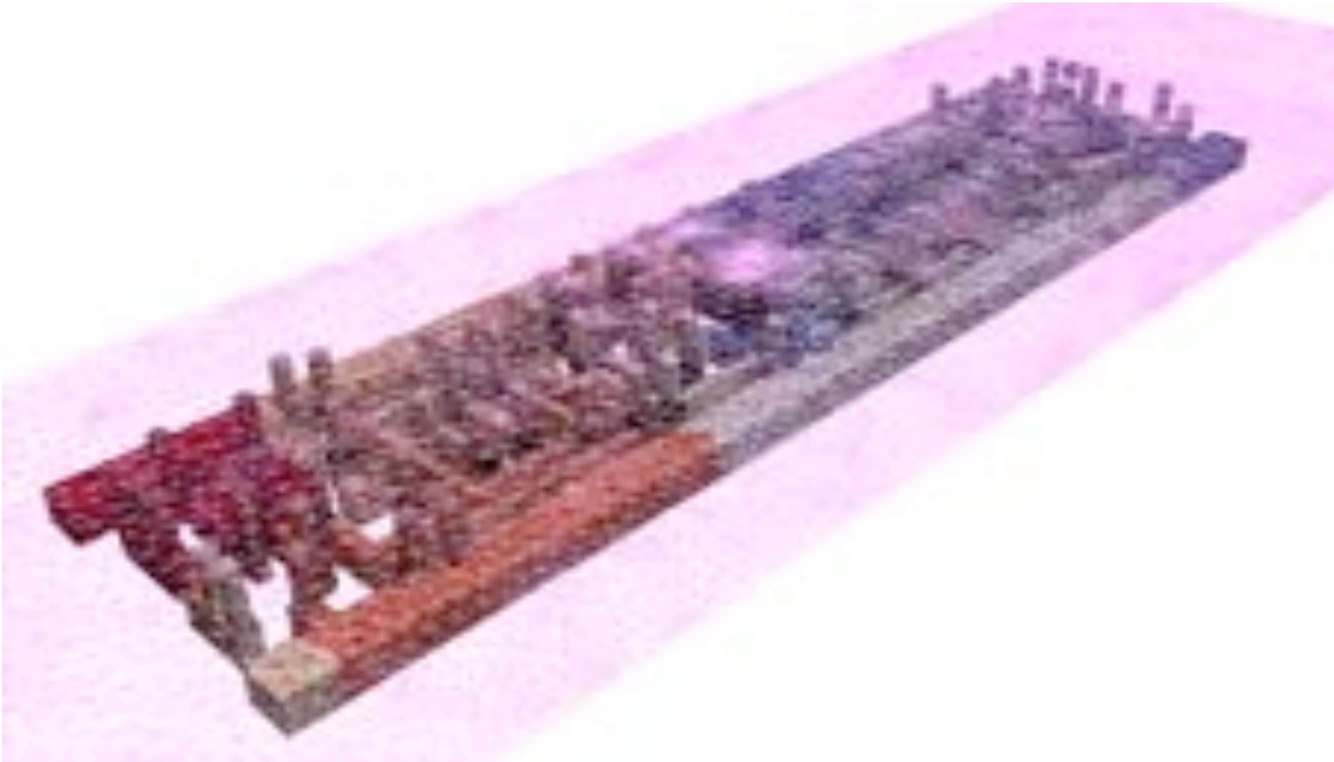
Theorem [Chen-C.-Cui-Zhang, 2010] We have

$$\|\nabla\times(\mathbf{A}-\mathbf{A}_h)\|_{L^2(\Omega)}+\alpha\|\,\mathbf{A}-\mathbf{A}_h\,\|_{L^2(\Omega_c)}\le C\min(1,\alpha)^{-1}\Big(\sum_{T\in\mathcal{M}_h}\eta_T^2\Big)^{1/2},$$

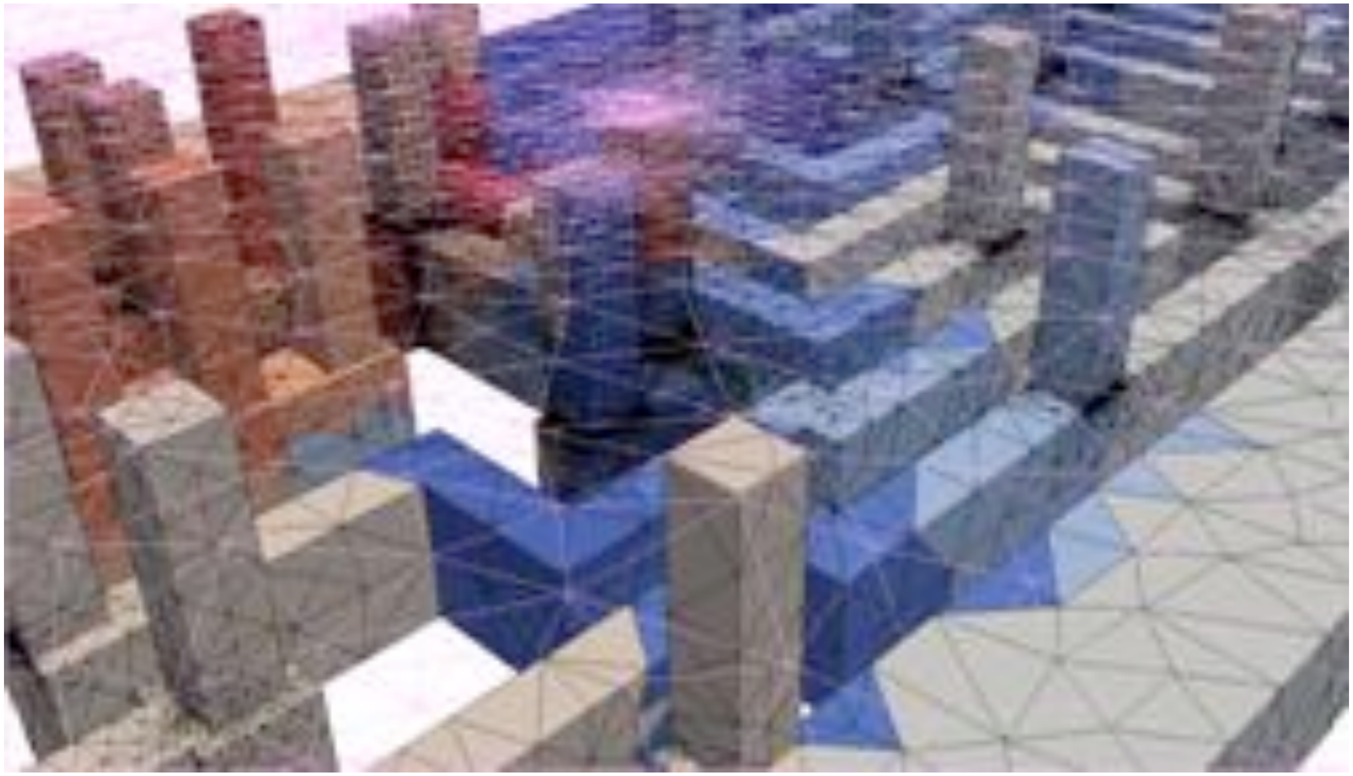
= where $\alpha=\sqrt{s^2\omega\sigma\mu}$ and, for any $T\in\mathcal{M}_h$,

$$\begin{aligned}
\eta_T^2 &= h_T^2\|\,s^2\mu\mathbf{J}_s-s^2\sigma\mu(s^{-1}\nabla\phi_0+\mathrm{i}\omega\mathbf{A}_h)\,\|_{L^2(T)}^2\\
&+ h_T^2\|\,s^2\mu\sigma\mathrm{div}(s^{-1}\nabla\phi_0+\mathrm{i}\omega\mathbf{A}_h)\,\|_{L^2(T)}^2\\
&+ \sum_{F\in\mathcal{F},F\subset\partial T}h_F\|\,[\![\nu\times\nabla\times\mathbf{A}_h]\!]_F\,\|_{L^2(F)}^2\\
&+ \sum_{F\in\mathcal{F},F\subset\partial T}h_F\|\,[\![s^2\sigma\mu(s^{-1}\nabla\phi_0+\mathrm{i}\omega\mathbf{A}_h)\cdot\nu]\!]_F\,\|_{L^2(F)}^2.
\end{aligned}$$

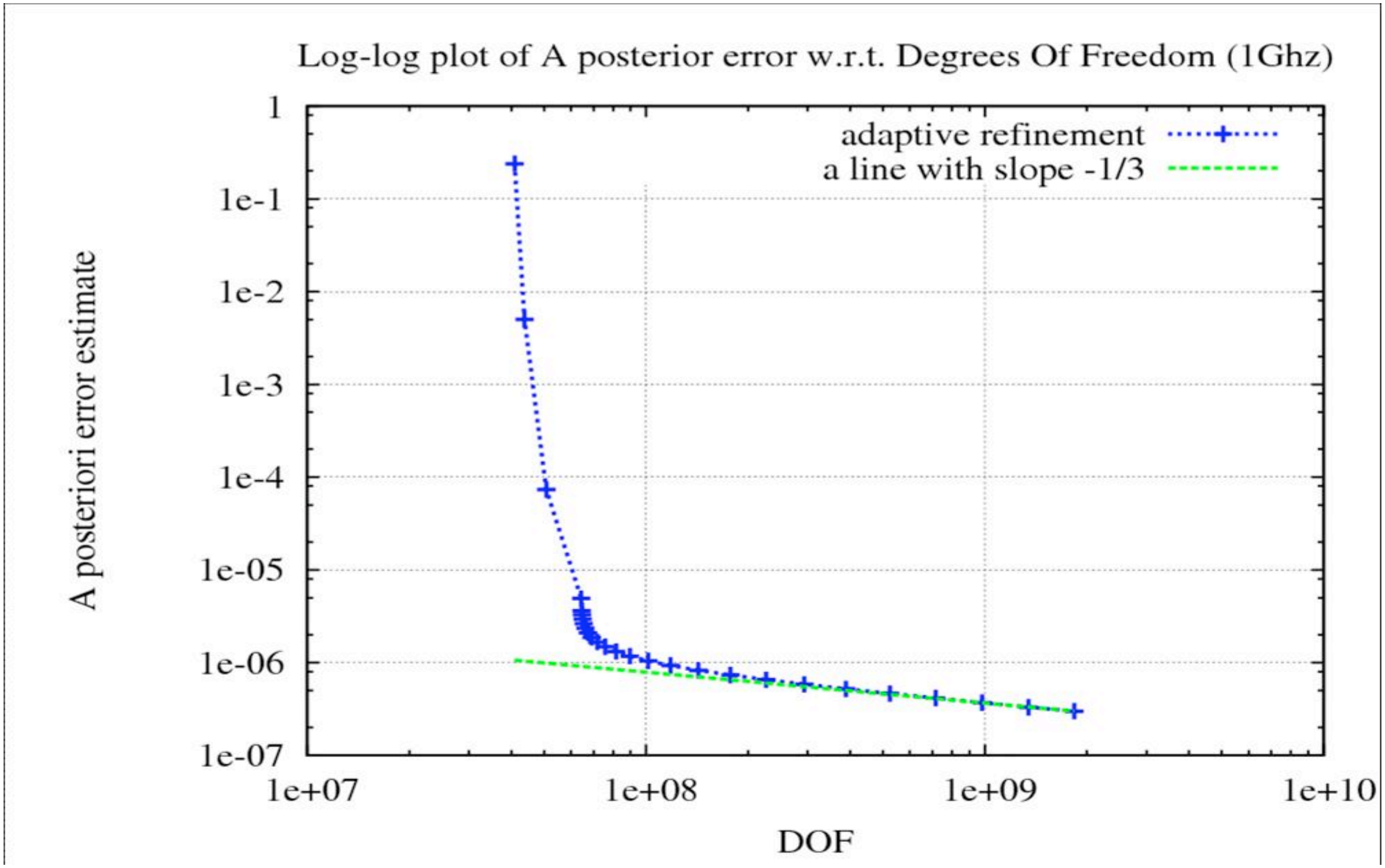
Numerical example



The structure of the "circuit of addition" (1 billion elements).
By Tao Cui (LSEC) and Hengliang Zhu, Xuan Zeng (Fudan University).



The adaptively refined mesh of the "circuit of addition".



The quasi-optimality of the adaptive finite element method.

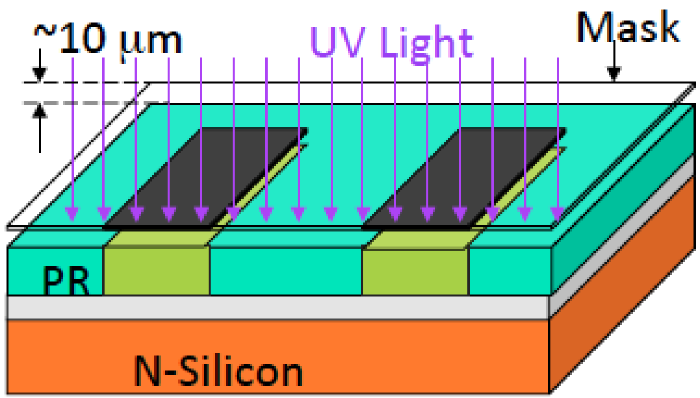
Contents

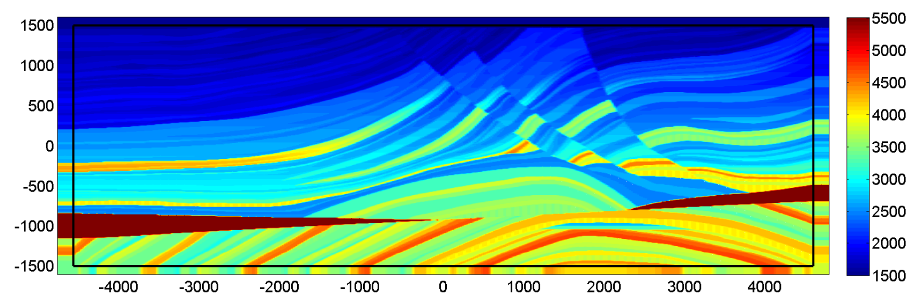
The time-harmonic waves

$$\Delta u + k(x)^2 u = f(x) \quad \text{in } \mathbb{R}^d \backslash \bar{D}, \quad d = 1, 2, 3,$$

$$u = g \quad \text{on } \Gamma_D, \quad r^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial r} - \mathbf{i} k u \right) \to 0 \quad \text{as } r = |x| \to \infty.$$

Here $k(x) = \omega / c(x)$ is constant outside some compact set; $f(x)$ is compactly supported.





The Sommerfeld radiation condition

$$\Delta u + k^2 u = f \quad \text{in } \mathbb{R}^3$$

Fundamental solution: $G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$,

$$\Delta G(x,y) + k^2 G(x,y) = -\delta_y(x) \quad \text{in } \mathbb{R}^3;$$

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Integral representation: let $\text{supp}(f) \subset B_R$, **for** $x \in B_R$,

$$u(x) = -\int_{B_R} G(x,y) f(y) dy + \int_{\partial B_R} \left(\frac{\partial u(y)}{\partial r(y)} G(x,y) - \frac{\partial G(x,y)}{\partial r(y)} u(y) \right) dy.$$

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Radiation condition: The integral on ∂B_R should vanish when $R \to \infty$.

$$\lim_{R\to\infty}\int_{\partial B_R}\left(\frac{\partial u(y)}{\partial r(y)}G(x,y)-\frac{\partial G(x,y)}{\partial r(y)}u(y)\right)dy=0.$$

It is obvious $G(x,y)=O(r(y)^{-1}), \frac{\partial G(x,y)}{\partial r(y)}=O(r(y)^{-1}), r(y)=|y|$. **However, the integral does not vanish if we impose**

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Observing that $\frac{\partial G(x,y)}{\partial r(y)}-\mathrm{i}kG(x,y)=O(r(y)^{-2})$, **[Sommerfeld, 1898] imposed instead,**

$$u\to 0,\quad r\left(\frac{\partial u}{\partial r}-\mathrm{i}ku\right)\to 0\ \text{ as } r\to\infty.$$

Under the Sommerfeld radiation condition, we have the existence and uniqueness

$$u(x) = -\int_{\mathbb{R}^3} G(x,y) f(y) dy.$$

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The above method does not apply to the general problem with inhomogeneous $k(x)$ or with the scatterer $D \neq \emptyset$.

Uniqueness: [Rellich 1943]
Existence: [Colton-Kress, 1983], [McLean, 2000] (method of BIE).

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Uniqueness: [Rellich 1943]
Existence: [Colton-Kress, 1983], [McLean, 2000] (method of BIE).

"By uniqueness ... we may be convinced that the unique solution of the mathematical problem is identical to the solution that is realized in nature." [Sommerfeld, 1949]

The limiting absorption principle

Let $u_\varepsilon \in H^1(\mathbb{R}^d)$, $\varepsilon > 0$, be the solution of

$$\Delta u_\varepsilon + (1 + \mathrm{i}\varepsilon)k(x)^2 u_\varepsilon = f \quad \text{in } \mathbb{R}^d, \qquad u_\varepsilon = g \quad \text{on } \Gamma_D.$$

It can be proved

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^d \setminus \bar{D})} \leq C(\|f\|_{H^1(\mathbb{R}^d)'} + \|g\|_{H^{1/2}(\Gamma_D)}), \quad s > 1/2.$$

where for any $s \in \mathbb{R}$, $\|v\|_{H^{1,s}(\mathcal{D})} = \left(\|v\|^2_{L^{2,s}(\mathcal{D})} + \|\nabla v\|^2_{L^{2,s}(\mathcal{D})}\right)^{1/2}$ and $\|v\|_{L^{2,s}(\mathcal{D})} = \left(\int_{\mathcal{D}} (1 + |x|^2)^s |v|^2 dx\right)^{1/2}$. The existence of the scattering solution of the Helmholtz equation can be proved by letting $\varepsilon \to 0$.

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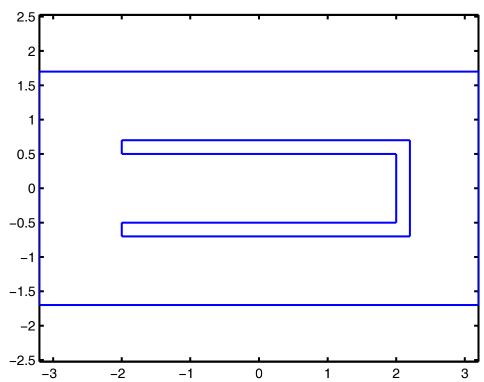
Open questions: layered media, periodic media, ...

The absorbing boundary condition

$$\Delta u + k(x)^2 u = f(x) \quad \text{in } \mathbb{R}^2 \backslash \bar{D},$$

$$u = g \quad \text{on } \Gamma_D, \qquad \sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty.$$

This problem is reduced to the problem in bounded domain by the DtN mapping $\mathbb{T} : H^{1/2}(\Gamma_l) \to H^{-1/2}(\Gamma_l)$, $B_l = \{x \in \mathbb{R}^2 : |x_j| \le l_j, j = 1, 2\}$,



$$\Delta u + k(x)^2 u = f(x) \quad \text{in } B_l \backslash \bar{D},$$

$$u = g \ \ \text{on } \Gamma_D,$$

$$\frac{\partial u}{\partial \nu} = \mathbb{T}u \ \ \text{on } \Gamma_l.$$

[Engquist-Majda, 1977] introduced a systematic way to construct the approximation of the DtN mapping. ABC is related to the wave-extrapolation equation in seismic imaging to find the down-going waves.

[Claebout-Muir, 1973] inserts the plane wave $e^{-\mathrm{i}\omega t+\mathrm{i}k_x x+\mathrm{i}k_z z}$ into the wave equation to obtain the dispersion relation $k_x^2+k_z^2=\omega^2/c^2=k^2$. By down-going wave one sloves

$$\mathrm{i}k_z=\mathrm{i}k\sqrt{1-\frac{k_x^2}{k^2}}\approx \mathrm{i}k\left(1-\frac{1}{2}\frac{k_x^2}{k^2}\right), \quad \text{for } \frac{k_x}{k}\ll 1.$$

Taking the inverse Fourier transform $\mathrm{i}k_x\leftrightarrow\frac{\partial}{\partial x}$, $\mathrm{i}k_z\leftrightarrow\frac{\partial}{\partial z}$, one obtains

$$\frac{\partial u}{\partial z}=\mathrm{i}ku+\frac{\mathrm{i}}{2k}\frac{\partial^2 u}{\partial x^2}.$$

This is the so called 15° equation.

The perfectly matched layer method

Helmholtz equation in the unbounded domain:

$$\Delta u + k^2 u = f \quad \text{in } \mathbb{R}^2 \,, \qquad |x|^{1/2}\Big(\frac{\partial u}{\partial |x|} - \mathrm{i}ku\Big) \to 0 \quad \text{as } |x| \to \infty,$$

where k is the wave number and $f \in H^{-1}(\mathbb{R}^2)$ has compact support.

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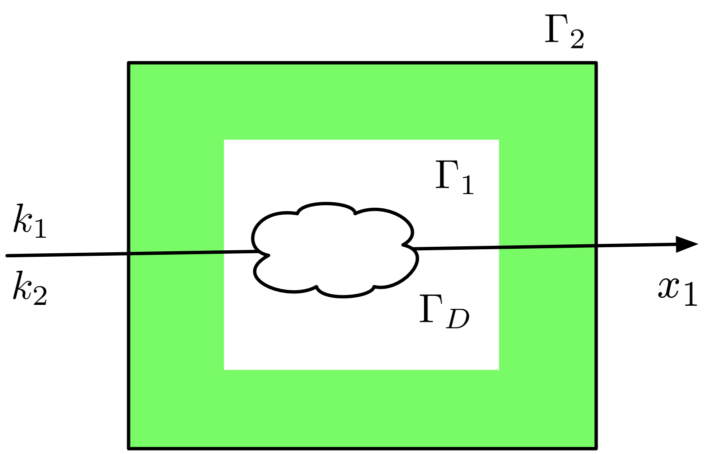

- **Perfectly Matched Layer (PML) [Berenger,1994]**
- **Uniaxial PML [Sacks-Kingsland-Lee-Lee, 1995]**

The complex coordinate stretching

Let $B_l = \{x \in \mathbb{R}^2 : |x_j| \le l_j, j = 1, 2\}$. Plane wave: $u = e^{\mathrm{i}kx_j}$, $j = 1, 2$, [Chew-Weedon, 1994] defines

$$x_j \;\to\; \tilde{x}_j = \int_0^{x_j} \alpha_j(t)dt = \begin{cases} x_j & \text{if } x_j \le l_j; \\ x_j + \mathrm{i} \int_{l_j}^{x_j} \sigma_j(t)dt & \text{if } x_j \ge l_j, \end{cases}$$

where $\alpha_j(t) = 1 + \mathrm{i}\sigma_j(t)$, $\sigma_j(t) = 0$ for $|t| \le l_j$,

$$\sigma_j(t) = \sigma_j(-t), \quad \text{and } \sigma_j = \gamma_0 > 0 \text{ for } |t| \ge \bar{l}_j.$$

Here $\bar{l}_j > l_j$ is fixed and $\gamma_0 > 0$ is a constant. The stretched wave:

$$\tilde{u} = e^{\mathrm{i}k\tilde{x}_j} = \begin{cases} e^{\mathrm{i}kx_j} & \text{if } x_j \le l_j; \\ e^{\mathrm{i}kx_j} e^{-k\int_{l_j}^{x_j} \sigma_j(t)dt} & \text{if } x_j \ge l_j. \end{cases}$$

The fundamental solution in $\mathbb{R}^2$

$$\Delta G(x,y) + k^2 G(x,y) = -\delta_y(x) \quad \text{in } \mathbb{R}^2.$$

We know $G(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|)$. Let

$$\hat{G}(\xi, x_2) = \int_{-\infty}^{\infty} G(x_1, x_2) e^{-\mathrm{i}(x_1 - y_1)\xi} dx_1$$

be the Fourier transform in first variable.

$$\frac{\partial^2 \hat{G}}{\partial x_2^2} + (k^2 - \xi^2)\hat{G} = -\delta_{y_2}(x_2).$$

Let $\mu = (k^2 - \xi^2)^{1/2}$, then $\hat{G}(\xi, x_2) = \frac{\mathrm{i}}{2\mu} e^{\mathrm{i}\mu|x_2 - y_2|}$.

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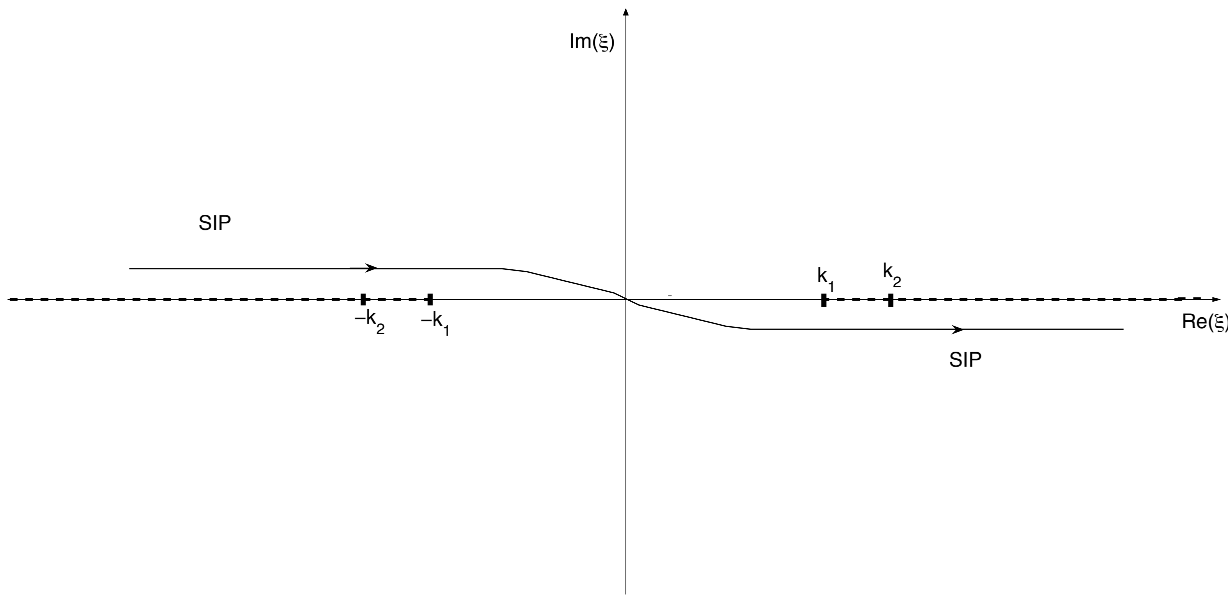
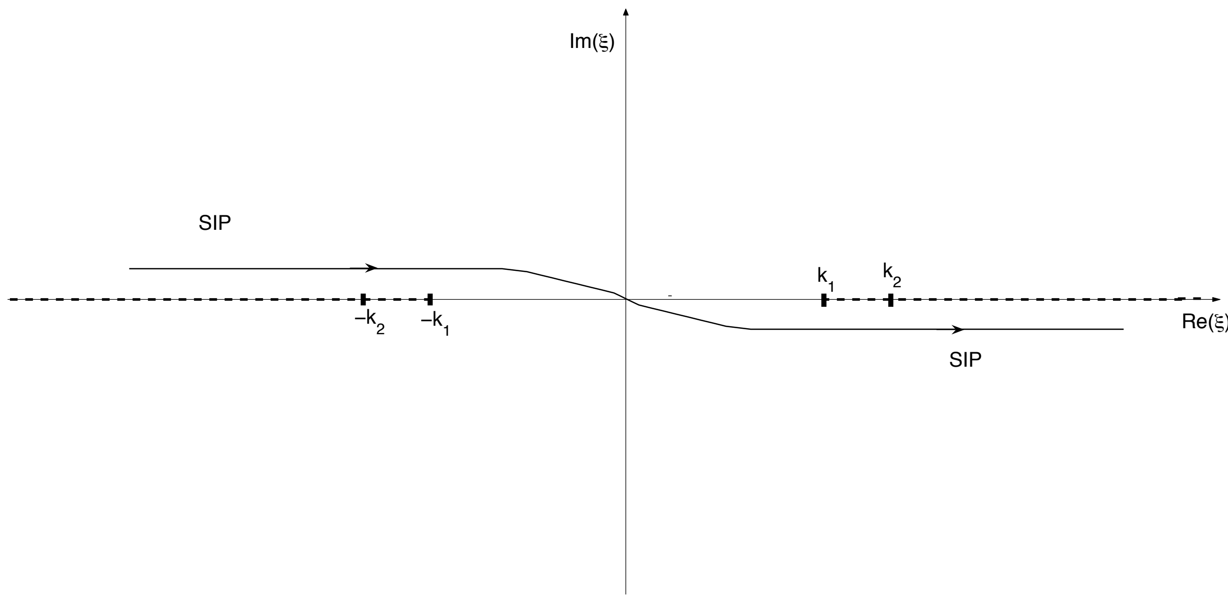


The Sommerfeld Integral Path (SIP)

Let $\mu = (k^2 - \xi^2)^{1/2}$, then $\hat{G}(\xi, x_2) = \frac{\mathrm{i}}{2\mu} e^{\mathrm{i}\mu|x_2 - y_2|}$.



The Sommerfeld Integral Path (SIP)

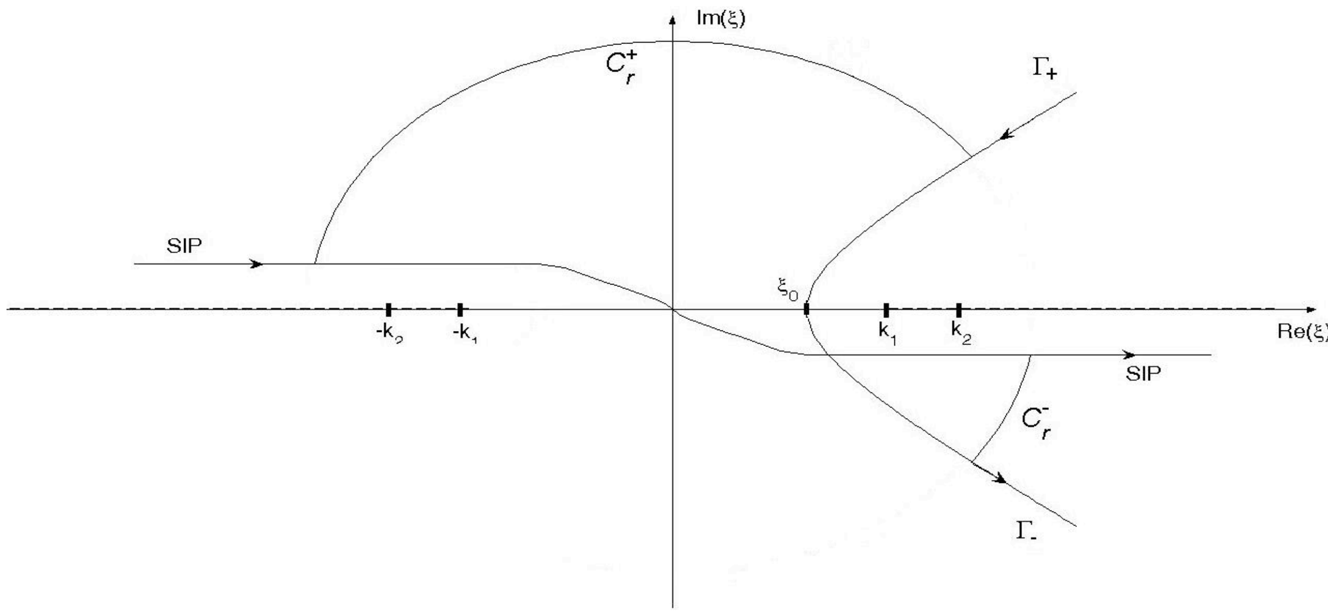
The Green function $G(x, y)$ is

$$G(x, y) = \frac{\mathrm{i}}{4\pi} \int_{\mathrm{SIP}} \frac{1}{\mu} e^{\mathrm{i}\xi(x_1 - y_1) + \mathrm{i}\mu(x_2 + y_2)} d\xi,$$

Cagniard-de Hoop transform

Lemma (Schläfli integral representation) We have

$$G(x,y) = \frac{1}{2\pi}\int_1^\infty \frac{1}{\sqrt{t^2-1}} e^{\mathrm{i}k|x-y|t}dt.$$



The convergence of the PML method

$$u(x) = -\int_{\mathbb{R}^2} f(y)G(x,y)dy \quad \text{in } \mathbb{R}^2.$$

For any $z \in \mathbb{C}$, denote by $z^{1/2}$ the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash[0,+\infty)$. We define the complexified distance

$$\rho(\tilde{x},\tilde{y}) = \left[(\tilde{x}(x_1) - \tilde{y}(y_1))^2 + (\tilde{x}(x_2) - \tilde{y}(y_2))^2\right]^{1/2},$$

and the complex coordinate stretched fundamental solution

$$G(\tilde{x},\tilde{y}) = \frac{1}{2\pi}\int_1^\infty \frac{1}{\sqrt{t^2-1}} e^{\mathrm{i}k\rho(\tilde{x},\tilde{y})t}dt.$$

The complexified distance satisfies, where $\sigma_0 = \max_{x\in\mathbb{R}} |\sigma(t)|$.

$$(1+\max(1,\sigma_0)^2)^{-1/2}|x-y| \le |\rho(\tilde{x},\tilde{y})| \le (1+\sigma_0^2)^{1/2}|x-y|, \quad \forall x,y\in\mathbb{R}^2.$$

The imaginary part of the complexified distance satisfies

$$\mathrm{Im}\,\rho(\tilde{x},\tilde{y}) \ge \frac{1}{|x-y|}\left(|x_1-y_1|\left|\int_{y_1}^{x_1}\sigma_1(t)dt\right| + |x_2-y_2|\left|\int_{y_2}^{x_2}\sigma_2(t)dt\right|\right).$$

There exists a constant $C>0$ independent of k and the medium property σ such that for any $x,y\in\mathbb{R}^2, x\neq y$,

$$|G(\tilde{x},\tilde{y})| \le C(1+\sigma_0^2)^{1/4} e^{-\frac{1}{2}k\,\mathrm{Im}\,\rho(\tilde{x},\tilde{y})}(k|x-y|)^{-1/2}.$$

Thus it decays exponentially as $|x|\to\infty$ for any fixed $y\in\mathbb{R}^2$.

$$\tilde{u}(x) = -\int_{\mathbb{R}^2} f(y) G(\tilde{x}, \tilde{y}) dy, \quad \forall x \in \mathbb{R}^2.$$

Since f is supported inside B_l, $\tilde{u} = u$ in B_l, $\tilde{u}$ decays exponentially as $|x| \to \infty$. $\tilde{u}$ satisfies $\tilde{\Delta}\tilde{u} + k^2\tilde{u} = f$ in $\mathbb{R}^2$ and by the chain rule

$$J^{-1}\nabla \cdot (A\nabla \tilde{u}) + k^2 \tilde{u} = f \quad \text{in } \mathbb{R}^2,$$

where $A(x) = \mathrm{diag}\left(\frac{\alpha_2(x_2)}{\alpha_1(x_1)}, \frac{\alpha_1(x_1)}{\alpha_2(x_2)}\right)$ and $J(x) = \alpha_1(x_1)\alpha_2(x_2)$.

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Let $B_L = \{x \in \mathbb{R}^2 : |x_j| \le l_j + d_j, j = 1, 2\}$. The PML problem is to find $\hat{u} \in H_0^1(B_L)$ such that

$$(A\nabla\hat{u}, \nabla\psi) - k^2(J\hat{u}, \psi) = -(Jf, \psi), \quad \forall \psi \in H_0^1(B_L).$$

Theorem [C.-Xiang, 2013] For sufficiently large $\sigma_0 d_2 \geq 1$, the sesquilinear form associated with the PML problem satisfies the inf-sup condition

$$\sup_{\psi \in H_0^1(B_L)} \frac{|(A\nabla\phi, \nabla\psi) - k^2(J\phi, \psi)|}{\|\psi\|_{H^1(B_L)}} \geq \mu_L \|\phi\|_{H^1(B_L)} \quad \forall \phi \in H_0^1(B_L),$$

where the constant $\mu_L^{-1} \leq Ck^{3/2}$. Moreover, we have

$$\|\tilde{u} - \hat{u}\|_{H^1(B_L)} \leq Ck^2(1 + kL)^2 e^{-\frac{1}{2}k\gamma\bar{\sigma}} \|f\|_{H^1(B_l)'},$$

where L is the diameter of B_L and $\gamma = \frac{d_2}{\sqrt{d_2^2 + (2l_2 + d_1 + d_2)^2}}$.

The stability is proved by the Bramble-Pasicak reflection argument [Bramble-Pasciak, 2012].

The fundamental solution of the PML equation

The fundamental solution of the PML equation is $\tilde{G}(x,y) = J(y)G(\tilde{x},\tilde{y})$ [Lassas-Sommersalo, 2001], [Kim-Pasciak, 2010]:

$$J^{-1}\nabla\cdot(A\nabla\tilde{G}(x,y)) + k^2\tilde{G}(x,y) = -\delta_y(x) \ \text{ in } \mathbb{R}^2.$$

Notice that $\tilde{G}(x,y) \neq \tilde{G}(y,x)$. $\tilde{G}(y,x)$ is the fundamental solution of the adjoint equation:

$$\nabla\cdot(A\nabla(J^{-1}\tilde{G}(y,x))) + k^2\tilde{G}(y,x) = -\delta_y(x) \ \text{ in } \mathbb{R}^2.$$

We recall that $\nabla\cdot A\nabla J^{-1}$ is the adjoint operator of $J^{-1}\nabla\cdot A\nabla$.

Bibliographic remarks (Convergence of PML method)

- **Convergence for circular or smooth PML layers**
  - **[Lassas-Somersalo, 1998, 2001]: Helmholtz**
  - **[Hohage-Schmidt-Zschiedrich, 2003]: Helmholtz**
  - **[C.-Liu, 2005]: 2D Helmholtz**
  - **[Bao-Wu, 2005]: Maxwell**
  - **[Chen-C, 2006] Maxwell**
  - **[Bramble-Pasciak, 2007, 2010]: Maxwell, elastic**
  - **[C.-Zheng, 2017] 3D Maxwell (two-layered)**

- **Convergence of Cartesian PML method**
  - **[C.-Wu, 2003]: grating problem**
  - **[C.-Zheng, 2010]: 2D Helmholtz (two-layered)**
  - **[Kim-Pasciak, 2010]: 2D Helmholtz**

- **[Bramble-Pasciak, 2012, 2013]: Helmholtz and Maxwell**
- **[C.-Cui-Zhang, 2013]: Maxwell (anisotropic PML)**
- **[C.-Xiang-Zhang, 2016]: elastic**

- **[Bramble-Pasciak, 2012, 2013]: Helmholtz and Maxwell**
- **[C.-Cui-Zhang, 2013]: Maxwell (anisotropic PML)**
- **[C.-Xiang-Zhang, 2016]: elastic**

Open questions: layered media, time-domain PML, ...

The discrete Helmholtz equation

The difficulties of solving large wave number discrete Helmholtz equations:

1. **Huge number of degrees of freedom required;**
2. **Highly indefinite nature of the discrete problem.**

The discrete Helmholtz equation

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Literature remarks:

1. **Domain decomposition [Despres, 1991], [Dhaidurov-Ogorodnikov, 1991]; [Gander-Magoules-Nataf, 2002];**
2. **Multigrid [Brandt-Livshit, 1997], [Elman-Ernst-O'Leary, 2001];**
3. **Shifted Laplacian [Erlangga, 2008], [Gander-Graham-Spence, 2015];**
4. **Sweeping preconditioner with moving PML [Engquist-Ying, 2011]**
5. **Review [Gander-Zhang, 2018]**

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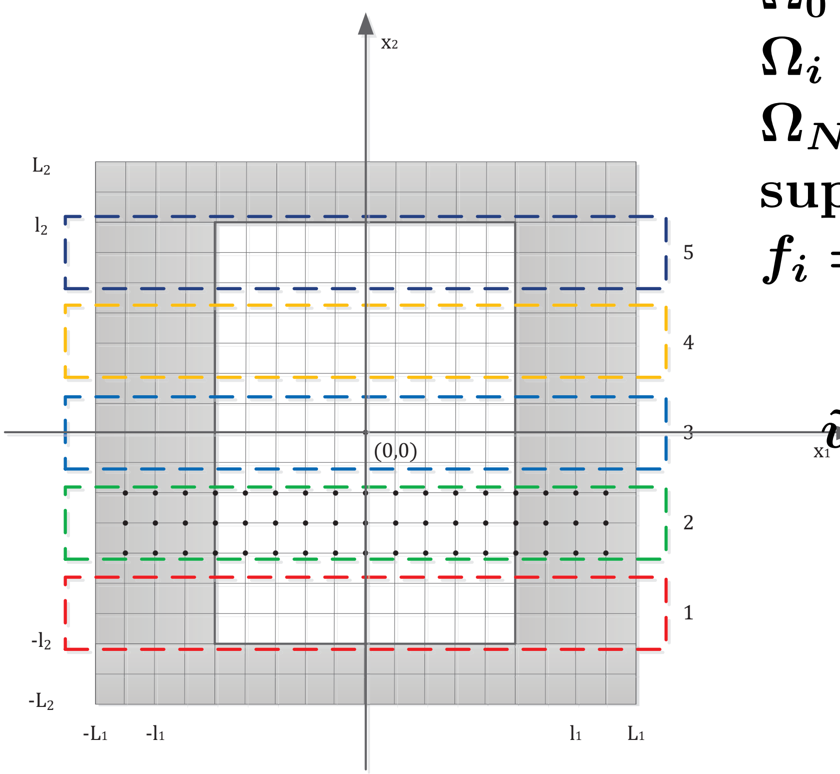
Our purpose: to propose a domain decomposition method:
Total costs $\propto$ # layers $\times$ the costs in each layer.

The STDDM: heuristic idea



$\Omega_0 = \{x \in \mathbb{R}^2 : x_2 < \zeta_1\}$
$\Omega_i = \{x \in \mathbb{R}^2 : \zeta_i < x_2 < \zeta_{i+1}\}$, $1 \le i \le N$,
$\Omega_{N+1} = \{x \in \mathbb{R}^2 : x_2 > \zeta_{N+1}\}$.
$\mathrm{supp}(f) \subset \cup_{i=1}^N \Omega_i$.
$f_i = f|_{\Omega_i}$ in Ω_i and 0 outside Ω_i.

$$\tilde{u}(x) = -\int_{\mathbb{R}^2} f(y)\tilde{G}(x,y)dy = -\sum_{i=1}^N \int_{\Omega_i} f_i(y)\tilde{G}(x,y)dy,$$

where $\tilde{G}(x,y)$ is the fundamental solution of the PML equation.

$$\tilde{u}(x) = -\int_{\mathbb{R}^2} f(y)\tilde{G}(x,y)dy = -\sum_{i=1}^{N}\int_{\Omega_i} f_i(y)\tilde{G}(x,y)dy,$$

Let $\bar{f}_1 = f_1$. We transfer the source from Ω_i to Ω_{i+1} in the sense that

$$\int_{\Omega_i} \bar{f}_i(y)\tilde{G}(x,y)dy = \int_{\Omega_{i+1}} [\Psi_{i+1}(\bar{f}_i)](y)\tilde{G}(x,y)dy, \quad \forall x \in \Omega_j, \quad j > i+1,$$

then for $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}(\bar{f}_i)$, we have

$$\tilde{u}(x) = -\int_{\Omega_N} f_N(y)\tilde{G}(x,y)dy - \int_{\Omega_{N-1}} \bar{f}_{N-1}(y)\tilde{G}(x,y)dy, \quad \forall x \in \Omega_N.$$

The solution $\tilde{u}$ in Ω_N can be solved outside only two layers Ω_N and Ω_{N-1}.

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The solution $\tilde{u}$ in Ω_N can be solved outside only two layers Ω_N and Ω_{N-1}. The solution in the other layers can be computed successively by solving the half-space Helmholtz problem using the transferred sources.

The STDDM for PML equation in $\mathbb{R}^2$

Algorithm 1. **(Source transfer for PML problem in $\mathbb{R}^2$)**
1° Let $\bar{f}_1 = f_1$ in $\mathbb{R}^2$;
2° For $i = 1, 2, \cdots, N-2$, do
- **Find $u_i \in H^1(\mathbb{R}^2)$ such that**

$$J^{-1}\nabla \cdot (A\nabla u_i) + k^2 u_i = -\bar{f}_i \quad \text{in } \mathbb{R}^2.$$

- **Compute $\Psi_{i+1}(\bar{f}_i) = J^{-1}\nabla \cdot (A\nabla(\beta_{i+1}u_i)) + k^2(\beta_{i+1}u_i)$ in Ω_{i+1}.**
- **Set $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}(\bar{f}_i)$ in Ω_{i+1} and $\bar{f}_{i+1} = 0$ elsewhere.**

Here $\beta_{i+1} = \beta_{i+1}(x_2)$ is a smooth function defined in Ω_{i+1},

$$\beta_{i+1} = 1, \beta'_{i+1} = 0 \quad \text{on } \Gamma_{i+1}, \quad \beta_{i+1} = \beta'_{i+1} = 0 \quad \text{on } \Gamma_{i+2},$$

$$\Gamma_i = \{x \in \mathbb{R}^2 : x_2 = \zeta_i\}.$$

Theorem [C.-Xiang, 2013] For $i = 1, \cdots, N-2$, we have, for any $x \in \Omega(\zeta_N, +\infty)$,

$$\int_{\Omega_i} \bar{f}_i(y)\tilde{G}(x,y)dy = \int_{\Omega_{i+1}} [\Psi_{i+1}(\bar{f}_i)](y)\tilde{G}(x,y)dy.$$

Remark: Other ways to transfer the source: [Stolk, 2013], [Zepeda-Nunez and Demanet, 2015]

Algorithm 2. (Wave expansion for PML problem in $\mathbb{R}^2$)
1° Solve $v_N \in H^1(\mathbb{R}^2)$ such that

$$J^{-1}\nabla \cdot (A\nabla v_N) + k^2 v_N = f_N + \bar{f}_{N-1} \quad \text{in } \mathbb{R}^2.$$

2° For $i = N-1, \cdots, 2$, find $v_i \in H^1(\Omega(-\infty, \zeta_{i+1}))$ such that

$$\begin{aligned} J^{-1}\nabla \cdot (A\nabla v_i) + k^2 v_i &= f_i + \bar{f}_{i-1} \quad \text{in } \Omega(-\infty, \zeta_{i+1}), \\ v_i &= v_{i+1} \quad \text{on } \Gamma_{i+1}. \end{aligned}$$

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Theorem [C.-Xiang, 2013] We have $\tilde{u} = v_N$ in $\Omega(\zeta_N, +\infty)$, $\tilde{u} = v_i$ in Ω_i for all $i = N-1, \cdots, 3$, and $\tilde{u} = v_2$ in $\Omega(-\infty, \zeta_3)$.

The source transfer algorithm in truncated domain

Algorithm 3. **(Source transfer for truncated PML problem)**
1° Let $\hat{f}_1 = f_1$ in Ω_1;
2° For $i = 1, \cdots, N-2$, do
• Find $\hat{u}_i \in H_0^1(\Omega_i^{\rm PML})$, $\Omega_i^{\rm PML} = (-l_1 - d_1, l_1 + d_1) \times (\zeta_i - d_2, \zeta_{i+2} + d_2)$, such that

$$(A_i \nabla \hat{u}_i, \nabla \psi) - k^2 (J_i \hat{u}_i, \psi) = (J_i \hat{f}_i, \psi), \quad \forall \psi \in H_0^1(\Omega_i^{\rm PML}).$$

• Compute $\hat{\Psi}_{i+1}(\hat{f}_i) \in H^{-1}(\Omega_i^{\rm PML})$ such that

$$\langle J_i \hat{\Psi}_{i+1}(\hat{f}_i), \psi \rangle = -(A_i \nabla(\beta_{i+1}\hat{u}_i), \nabla \psi) + k^2 (J_i \beta_{i+1} \hat{u}_i, \psi), \forall \psi \in H_0^1(\Omega_i^{\rm PML}).$$

• Set $\hat{f}_{i+1} = f_{i+1} + \hat{\Psi}_{i+1}(\hat{f}_i)$ in $\Omega_{i+1} \cap B_L$ and $\hat{f}_{i+1} = 0$ elsewhere.

The wave expansion algorithm in truncated domain

Algorithm 4. **(Wave expansion for truncated PML problem)**
1° **Solve** $\hat{v}_N$ **such that**

$$
\begin{aligned}
J_{N-1}^{-1}\nabla\cdot(A_{N-1}\nabla\hat{v}_N)+k^2\hat{v}_N &= f_N+\hat{f}_{N-1} \quad \text{in} \quad \Omega_{N-1}^{\mathrm{PML}},\\
\hat{v}_N &= 0 \quad \text{on} \quad \partial\Omega_{N-1}^{\mathrm{PML}}.
\end{aligned}
$$

2° **For** $i=N-1,\cdots,2$**, find** $\hat{v}_i$ **such that**

$$
\begin{aligned}
J_{i-1}^{-1}\nabla\cdot(A_{i-1}\nabla\hat{v}_i)+k^2\hat{v}_i &= f_i+\hat{f}_{i-1} \quad \text{in } D_i^{\mathrm{PML}},\\
\hat{v}_i &= \hat{v}_{i+1} \quad \text{on } \partial D_i^{\mathrm{PML}}\cap\Gamma_{i+1},\\
\hat{v}_i &= 0 \quad \text{on } \partial D_i^{\mathrm{PML}}\backslash\Gamma_{i+1},
\end{aligned}
$$

where $D_i^{\mathrm{PML}}=(-l_1-d_1,l_1+d_1)\times(\zeta_{i-1}-d_2,\zeta_{i+1})$.

Theorem [C.-Xiang, 2013] Let $\hat{v} = \hat{v}_N$ in $\Omega(\zeta_N, +\infty) \cap B_L$, $\hat{v} = \hat{v}_i$ in $\Omega_i \cap B_L$ for all $i = 3, \cdots, N-1$, and $\hat{v} = \hat{v}_2$ in $\Omega(-\infty, \zeta_2) \cap B_L$. Then for sufficiently large $\sigma_0 d_2 \geq 1$, we have

$$\|\hat{u} - \hat{v}\|_{H^1(B_L)} \leq Ck^{3N-\frac{7}{2}}(1+kL)^2 e^{-\frac{1}{2}k\gamma\bar{\sigma}}\|f\|_{H^1(B_l)'}.$$

Let T be the solution operator of the PML problem defined by $T(f) = \hat{u}$ and $\hat{T}$ be the output operator of the the STDDM defined by $\hat{T}(f) = \hat{v}$.

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Let T be the solution operator of the PML problem defined by $T(f) = \hat{u}$ and $\hat{T}$ be the output operator of the the STDDM defined by $\hat{T}(f) = \hat{v}$.

$\hat{T}$ is a good approximation of T if the PML parameters are chosen such that $k^{3N-\frac{7}{2}}(1 + kL)^2 e^{-\frac{1}{2}k\gamma\bar{\sigma}}$ is sufficiently small (e.g. 0.01 in our numerical examples).

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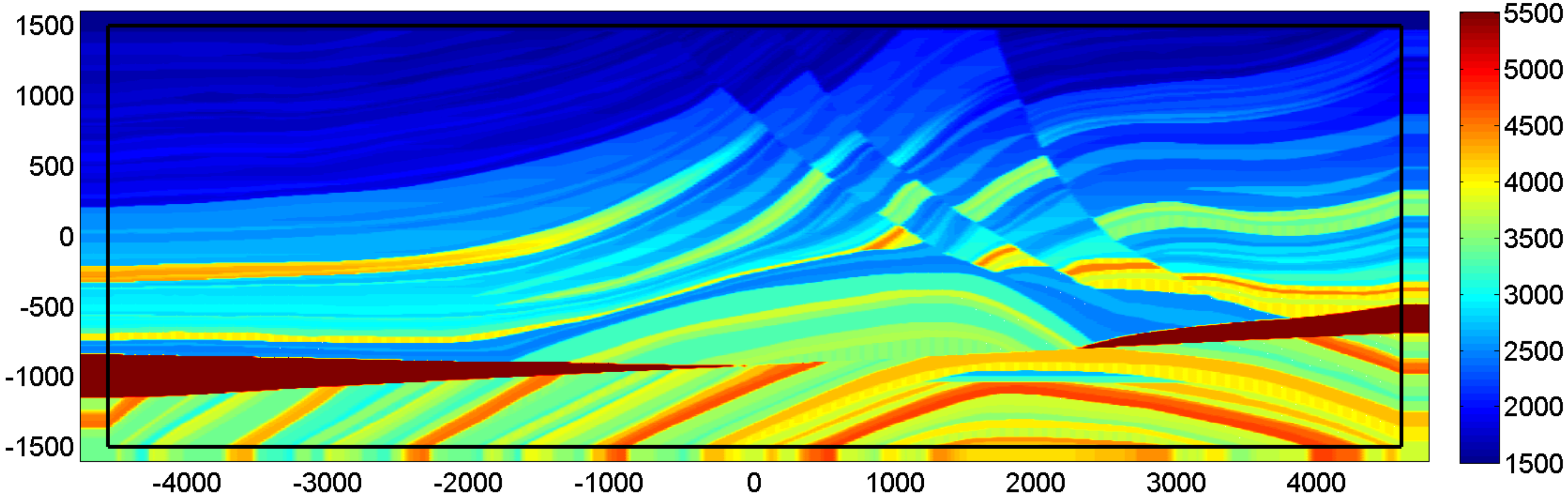
The discretization of $\hat{T}$ will be a good preconditioner of the corresponding discretization of T if the well-known pollution error of the discretization of the Helmholtz equation is controlled. Better discretization leads to better preconditioner.

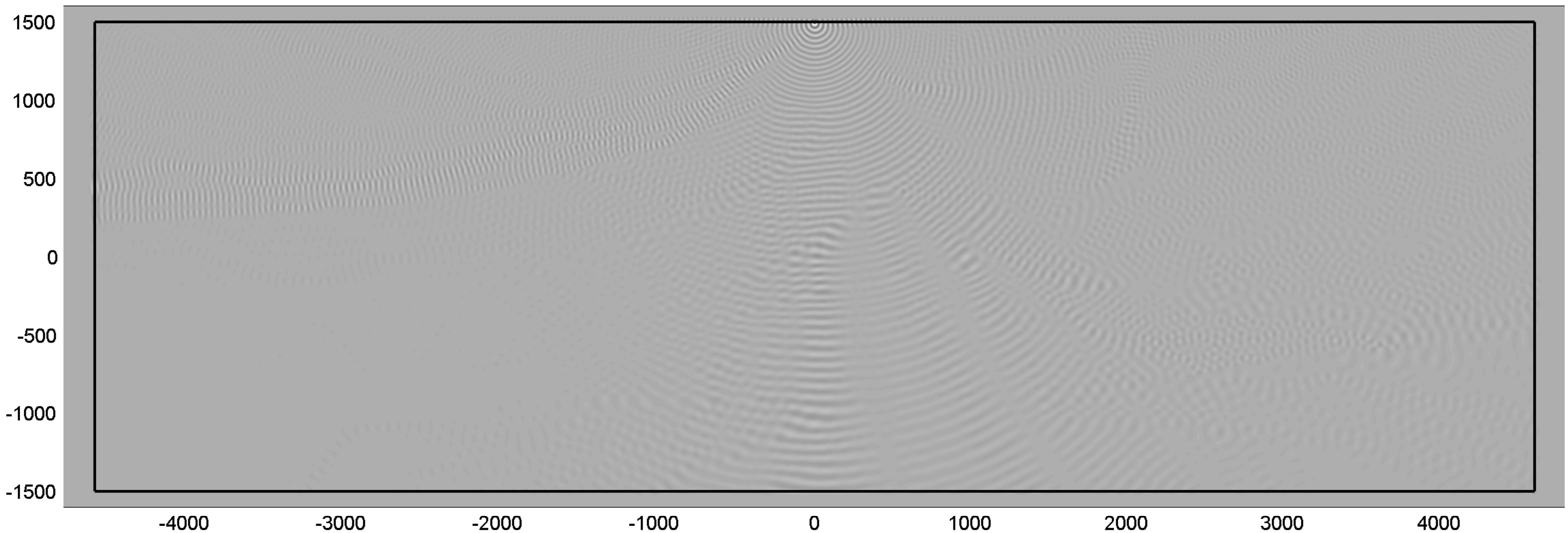
Numerical example : Helmholtz problem: Marmousi model

$k(x) = \omega/c(x)$, where $\omega/2\pi = 50$Hz and $c(x)$ is Marmousi model in $B_1 = (-4100m, 4100m) \times (-1500m, 1500m)$. Wavelength ranges in $30m - 110m$. There are 117×43 wavelengths in average. Let $N = 30$. The mesh $12.5m \times 4m$. Set $f(x) = e^{-(h_0^2/16)^{-2}((x_1-r_1)^2+(x_2-r_2)^2)}$.



Set $f(x) = e^{-(h_0^2/16)^{-2}((x_1-r_1)^2+(x_2-r_2)^2)}$. Spectral element of order p.

p	DOF	Number of iterations
3	5,534,305	76
5	15,367,841	65



The real part of the solution when $\omega/2\pi = 50$ and $p = 3$.

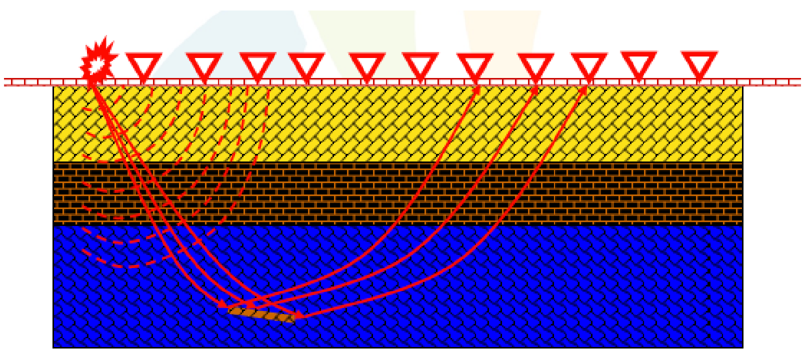
Contents

The setting

The forward problem:

$$\left(\frac{1}{c(x)^2}\frac{\partial^2}{\partial t^2} - \Delta\right) p = \delta_{x_s}(x) f(t) \quad \text{in } \mathbb{R}^3_+.$$

The data: $Q(x_s, x_r; t) = p(x_s, x_r; t)$, $x_s, x_r \in \Gamma_0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$.
The problem: From the data $Q(x_s, x_r, t)$ and a background velocity $c_0(x)$ to reconstruct the reflectivity $R(x)$.



The data fitting model

Let K be some closed convex set of $L^2(\mathbb{R}^3_+)$, e.g., $K = \{c \in L^2(\mathbb{R}^3_+) : 0 \leq c_1 \leq c(x) \leq c_2$ a.e. in $\mathbb{R}^3_+\}$.

Define the misfit functional

$$J(c) = \frac{1}{2}\sum_{j=1}^{N} \|p[c, x_{s_j}](\cdot) - Q(x_{s_j}, \cdot, t)\|^2_{L^2(\Gamma_0 \times (0,T))}.$$

The data fitting model:

$$\min_{c \in K} J(c).$$

Difficulties: nonlinear, non-convex, non-smooth, ill-posed, etc.

Reverse time migration method

The forward problem:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) p_F = \delta_{x_s}(x) f(t),$$
$$\partial_\nu p_F = 0 \quad \text{on } \Gamma_0,$$
$$p_F|_{t=0} = 0, \quad \partial_t p_F|_{t=0} = 0.$$

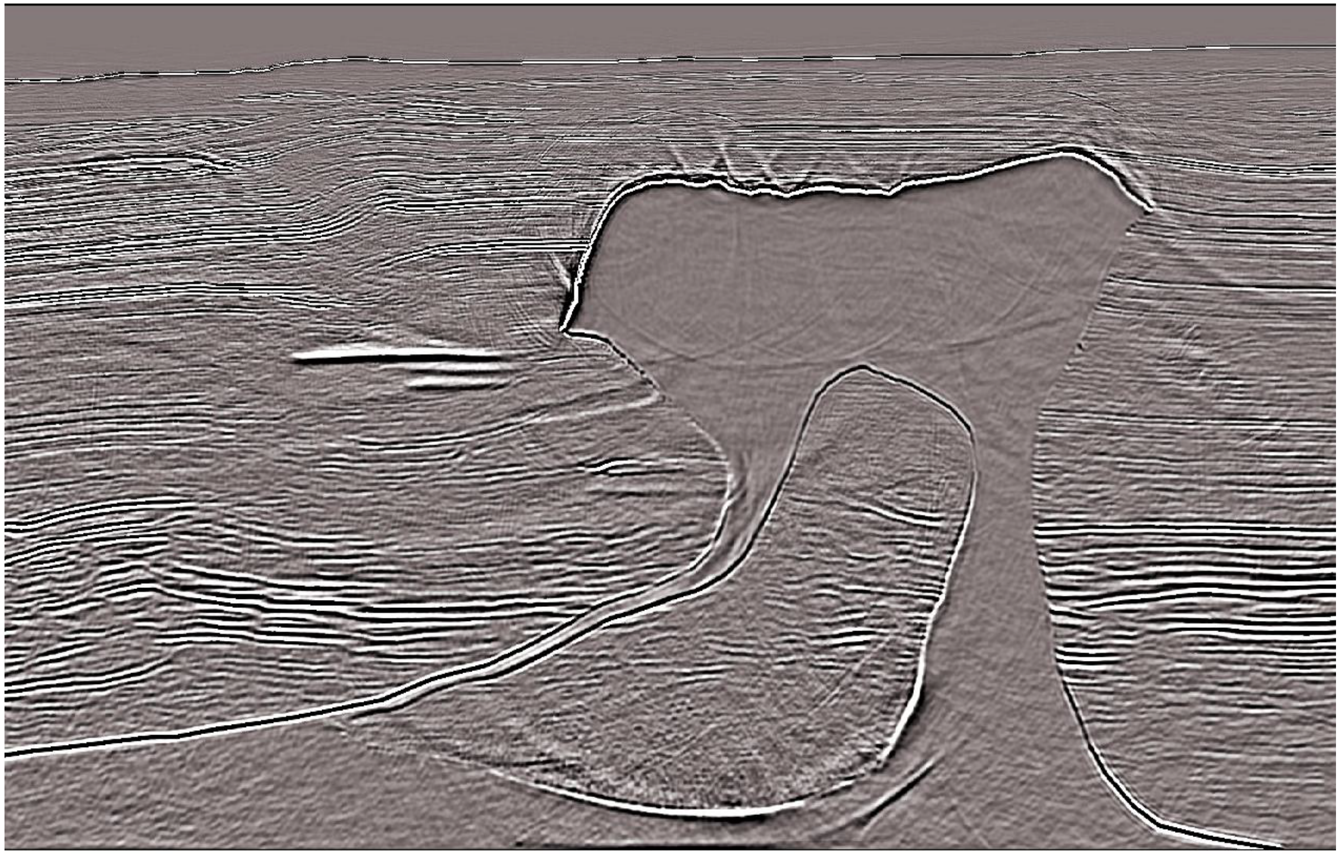
The back-propagation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) p_B = 0,$$
$$p_B = Q(x_s, x, t) \quad \text{on } \Gamma_0,$$
$$p_B|_{t=T} = 0, \quad \partial_t p_B|_{t=T} = 0.$$

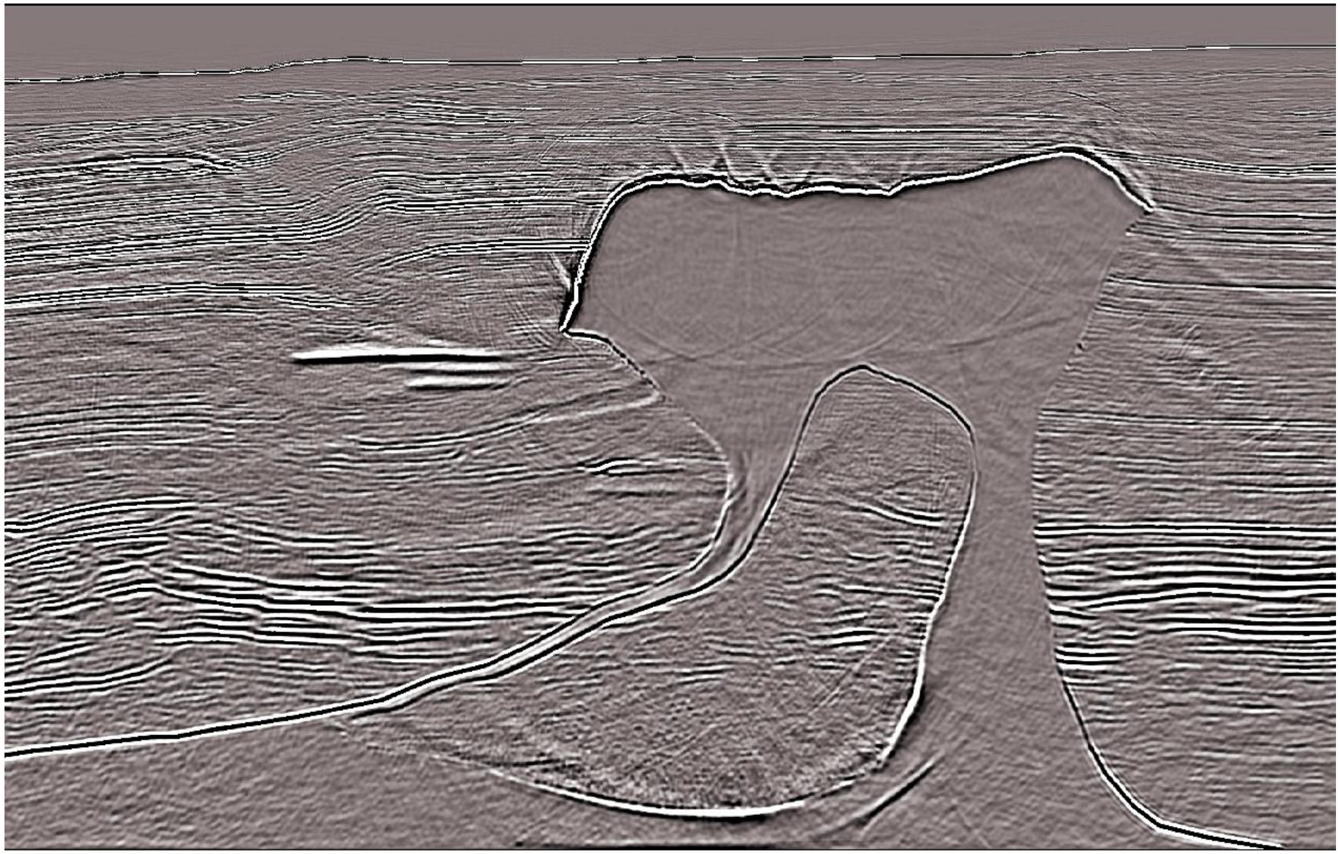
Imaging condition: [Baysal-Kosloff-Sherwood, 1983], [MaMechan, 1983], [Whitmore, 1983]

$$R(x) = \int_0^T \int_{\Gamma_0 \times \Gamma_0} p_F(x_s, x, t) p_B(x_s, x_r, x, t) ds(x_s) ds(x_r) dt.$$

Prestack depth migration: [Claerbout, 1970].



From Yu Zhang's LSEC summer school 2011.



From Yu Zhang's LSEC summer school 2011.

Our question: What is the image? Why RTM works?
Previous work: high frequency assumption

The acoustic scattering problem

Dirac sources $x_s \in \Gamma_s$.
Measurement: the scattered waves $u^s = u - u^i, u^i = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - x_s|)$.

The scattering problem of penetrable obstacles:

$$\Delta u + k^2 n(x) u = -\delta_{x_s}(x) \quad \text{in } \mathbb{R}^2,$$
$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \quad \text{as } r \to \infty, \quad r = |x|.$$

The scattering problem of non-penetrable obstacles:

$$\Delta u + k^2 u = -\delta_{x_s}(x) \quad \text{in } \mathbb{R}^2 \backslash \bar{D},$$
$$u = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} + \mathrm{i}k\eta(x) u = 0 \quad \text{on } \Gamma_D,$$
$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \quad \text{as } r \to \infty, \quad r = |x|.$$

Literature remarks

- **Multiple Signal Classification (MUSIC): point source or small inclusions [Schmidt, 1986], [Devaney, 2001], [Bruhl, Hanke and Vogelius, 2003], [Ammari and Kang, 2004].**
- **Linear Sampling [Colton and Kirsch, 1996], Factorization [Kirsch, 1998], Probe method [Ikehata,1998], and Point Source [Potthast, 1996].**
- **Reverse time migration (RTM): prestack depth migration [Claerbout, 1970].**

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- **Reverse time migration (RTM): prestack depth migration [Claerbout, 1970]. The analysis is based on small inclusion [Garnier, 2007], [Ammari, et al, 2013] or high frequency [Belykin, 1982], [Bleistein, Cohen, and Stockwell, 2001] assumptions.**

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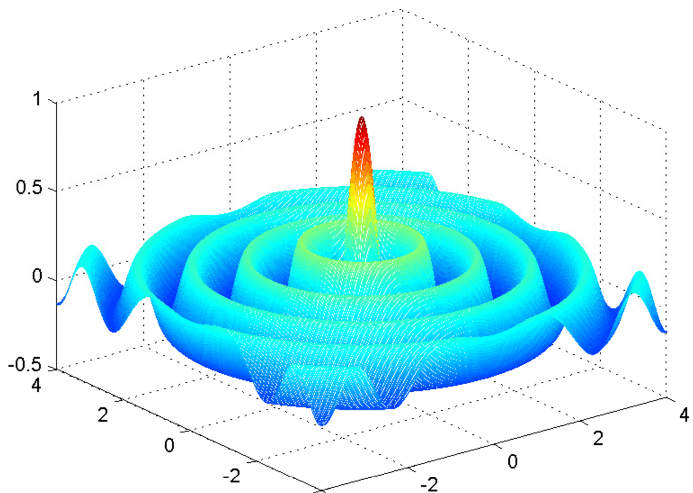
Our purpose: New mathematical understanding of RTM without the geometric optics approximation assumption for extended obstacles.

Helmholtz-Kirchhoff identity

Lemma [Bojarski, 1973] Let $\mathcal{D}$ be a bounded domain, for any $x, y \in \mathcal{D}$,

$$\int_{\partial\mathcal{D}} \left(\overline{\Phi(x,\xi)} \frac{\partial \Phi(\xi,y)}{\partial \nu} - \frac{\partial \overline{\Phi(x,\xi)}}{\partial \nu} \Phi(\xi,y) \right) ds(\xi) = 2\mathrm{i}\,\mathrm{Im}\Phi(x,y),$$

where $\Phi(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|)$. For $x \in \Omega, \xi \in \partial\mathcal{D}$, $\Phi(x,\xi) = O(R^{-1/2})$, $\frac{\partial\Phi(x,\xi)}{\partial\nu} - \mathrm{i}k\Phi(x,\xi) = O(R^{-3/2})$, where $R = \mathrm{dist}(\Omega, \partial\mathcal{D})$.



This implies, for $x, y \in \Omega$,

$$\begin{aligned} & k \int_{\partial\mathcal{D}} \Phi(x,\xi) \overline{\Phi(\xi,y)} ds(\xi) \\ = \;& \mathrm{Im}\,\Phi(x,y) + O(R^{-1}). \end{aligned}$$

The RTM algorithm

Given $u^s(x_r, x_s)$, where $x_r \in \Gamma_r = \partial B_r$, $x_s \in \Gamma_s = \partial B_s$.
1° Back-propagation: Compute the solution v_b:

$$\Delta v_b(x, x_s) + k^2 v_b(x, x_s) = \int_{\Gamma_r} \overline{u^s(x_r, x_s)} \delta_{x_r}(x) ds(x_r) \qquad \text{in } \mathbb{R}^2,$$

$$\sqrt{r}\left(\frac{\partial v_b}{\partial r} - \mathrm{i} k v_b\right) \to 0 \qquad \text{as } r \to \infty.$$

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2° Cross-correlation: For $z \in \Omega$, the search domain, compute

$$I(z) \;=\; k^2 \cdot \mathrm{Im} \int_{\Gamma_s} u^i(z, x_s) v_b(z, x_s) ds(x_s).$$

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$$\begin{aligned} I(z) &= k^2 \cdot \mathrm{Im} \int_{\Gamma_s} u^i(z, x_s) v_b(z, x_s) ds(x_s). \\ &= -k^2 \cdot \mathrm{Im} \int_{\Gamma_s} \int_{\Gamma_r} \Phi(z, x_s) \Phi(z, x_r) \overline{u^s(x_r, x_s)} ds(x_s) ds(x_r). \end{aligned}$$

The resolution analysis: sound soft obstacle

Theorem [Chen-C.-Huang, 2013] For any $z \in \Omega$, let $\psi(x,z)$ be the radiation solution of

$$\Delta\psi(x,z) + k^2\psi(x,z) = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{D}, \quad \psi(x,z) = -\mathrm{Im}\,\Phi(x,z) \quad \text{on } \Gamma_D.$$

Then we have

$$I(z) = k\int_{S^1} |\psi_\infty(\hat{x},z)|^2 d\hat{x} + w_{\hat{I}}(z)$$

where $\|w_{\hat{I}}\|_{L^\infty(\Omega)} \leq C(R_s^{-1} + R_r^{-1})$.

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$\psi(x,z)$ is the radiation solution of the Helmholtz equation with the incident wave $\mathrm{Im}\,\Phi(x,z)$. Therefore we expect that the imaging functional will peak at the boundary of the scatterer D and decay away from the scatterer.

Numerical examples

The synthesized scattering data is computed by standard Nyström's methods.

The boundary integral equations on Γ_D are solved on a uniform mesh of the boundary with ten points per probe wavelength.

The boundaries of the obstacles used in our numerical experiments are parameterized as follows:

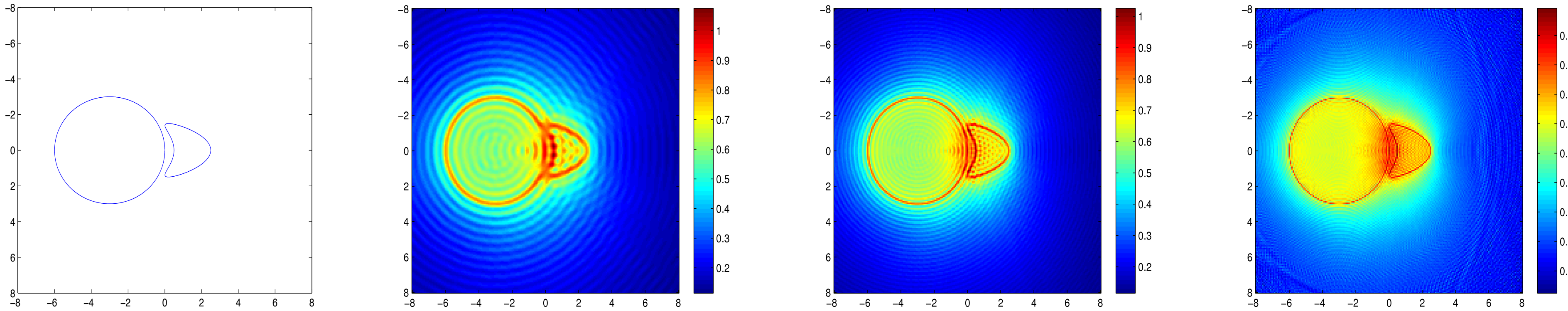
Circle: $x_1 = \rho\cos(\theta), \quad x_2 = \rho\sin(\theta), \quad \theta \in (0, 2\pi],$

Kite: $x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65, \quad x_2 = 1.5\sin(\theta), \quad \theta \in (0, 2\pi],$

p-leaf: $r(\theta) = 1 + 0.2\cos(p\theta), \quad \theta \in (0, 2\pi].$

The resolution of two obstacles

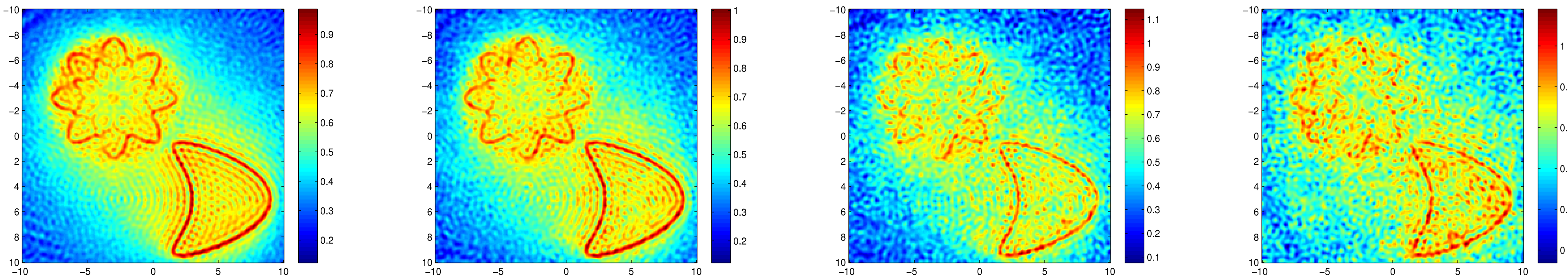
The model is a circle of radius $\rho = 3$ and a kite. The distance between two objects is about 0.5. The search domain is $\Omega = (-8, 8) \times (-8, 8)$ with a sampling 201×201 mesh.



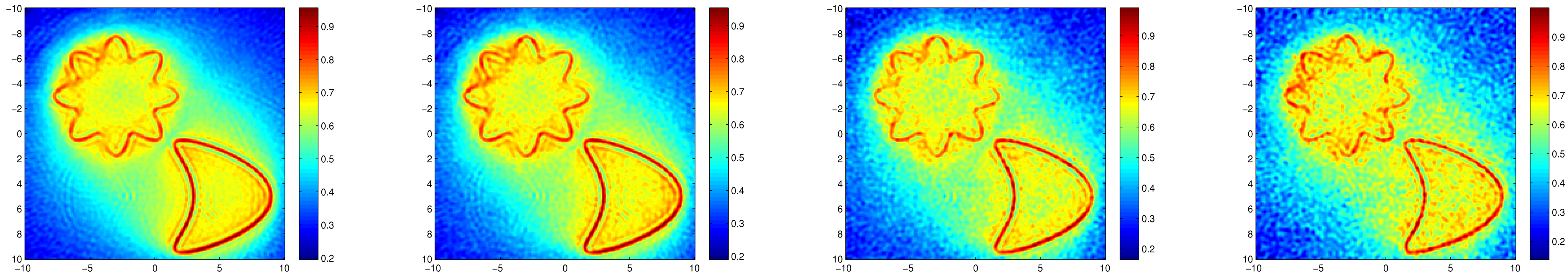
The first picture is the exact obstacles. The other three pictures from left to right are imaging results using probe wavelengths $\lambda = 1, 0.5, 0.25$ and $N_s = N_r = 64, 128, 256$.

Stability of RTM with respect to the additive Gaussian random noise

Let $u_{noise} = u_s + \nu_{noise}$, where $\nu_{\mathrm{noise}} \sim \mathcal{N}(0, \mu \max |u_s|)$ is the Gaussian noise.



The imaging results using data added with additive Gaussian noise and $\mu = 10\%, 20\%, 40\%, 60\%$ from left to right, respectively. The probe wavelength $\lambda = 1$ and $N_s = N_r = 128$. The search domain is $\Omega = (-10, 10) \times (-10, 10)$ with a sampling 201×201 mesh.



The imaging results using multi-frequency data added with additive Gaussian noise and $\mu = 10\%, 20\%, 40\%, 60\%$ from left to right, respectively. The probe wavelengths $\lambda = 1/0.8, 1/0.9, 1/1.0, 1/1.1, 1/1.2$ and $N_s = N_r = 128$.

Phaseless imaging

Let $u^i(x, x_s) = \Phi(x, x_s)$, where $\Phi(x, x_s) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - x_s|)$ and the source $x_s \in \Gamma_s$, be the incident wave. Let the total field is $u = u^i + u^s$ with $u^s(x, x_s)$ satisfying

$$
\begin{aligned}
&\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \\
&u^s = -u^i \quad \text{on } \Gamma_D, \\
&\sqrt{r}\left(\frac{\partial u^s}{\partial r} - \mathrm{i} k u^s\right) \to 0 \ \text{ as } \ r = |x| \to +\infty,
\end{aligned}
$$

The problem: Find the support of the scatterer using only the information $|u(x_r, x_s)|$.

Literature remarks

- **Uniqueness [Klibanov, 2014], [Novikov, 2015, 2016] (Schrödingier)**
- **Iterative method: [Litman-Belkebir, 2006], [D'Urso et al, 2008], [Li-Zheng-Li, 2009] Continuation: [Bao-Li-Lv, 2013], Linearization: [Ammari-Chow-Zou, 2016] , [Klibanov-Nguyen-Pan, 2015], [Klibanov-Romanov, 2015]; Modulus of Far field pattern [Kress-Rundell, 1997], [Ivanyshyn-Kress, 2001, 2010]**
- **Phase retrieval and imaging [Franceschini et al, 2006], [Bardsley-Vasquez, 2016], [Chai-Moscoso-Papanicolaou, 2010], [Novikov-Moscoso-Papanicolaou, 2015].**
- **Direct method: [Devaney, PRL 1989] (Born approximation)**

Phaseless imaging

The RTM imaging function:

$$I_{\mathrm{RTM}}(z) = -k^2 \mathrm{Im} \int_{\Gamma_s} \int_{\Gamma_r} \Phi(z, x_s)\Phi(x_r, z)\overline{u^s(x_r, x_s)} ds(x_r) ds(x_s).$$

Phaseless imaging function based on RTM:

$$I_{\mathrm{RTM}}^{\mathrm{phaseless}}(z) = -k^2 \mathrm{Im} \int_{\Gamma_s} \int_{\Gamma_r} \Phi(z, x_s)\Phi(x_r, z)\Delta(x_r, x_s) ds(x_r) ds(x_s),$$

where

$$\Delta(x_r, x_s) = \frac{|u(x_r, x_s)|^2 - |u^i(x_r, x_s)|^2}{u^i(x_r, x_s)}.$$

Phaseless imaging

Theorem [C.-Huang, 2017] We have

$$I_{\mathrm{RTM}}^{\mathrm{phaseless}}(z) = I_{\mathrm{RTM}}(z) + R_{\mathrm{RTM}}^{\mathrm{phaseless}}(z), \qquad \forall z \in \Omega,$$

where $|R_{\mathrm{RTM}}^{\mathrm{phaseless}}(z)| \leq C(1 + \|\mathbb{T}\|)^2(kR_s)^{-1/2}$. Here $\mathbb{T}$ is the DtN mapping and the constant C may depend on $kd_D, k|z|$ but is independent of k, d_D, R_r, R_s.

Therefore the imaging resolution of our new phaseless RTM algorithm is essentially the same as the imaging results using the scattering data with the full phase information when the sources and measurements are placed far away from the scatterer.

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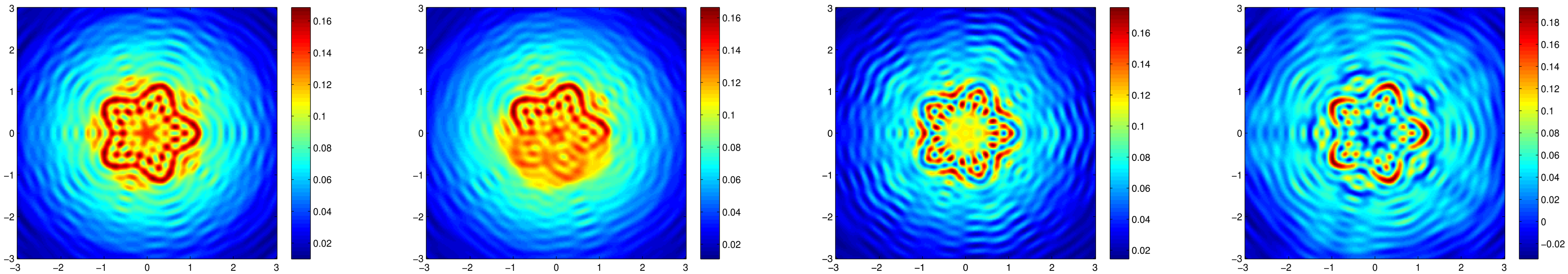
Therefore the imaging resolution of our new phaseless RTM algorithm is essentially the same as the imaging results using the scattering data with the full phase information when the sources and measurements are placed far away from the scatterer. The theorem extends to penetrable or other non-penetrable obstacles.

Numerical example

The imaging of a 5-leaf obstacle with impedance condition $\eta = 5$, a partially coated obstacle with $\eta = 5$ in the upper boundary and $\eta = 1$ in the lower boundary, a sound hard, and a penetrable obstacle with $n(x) = 0.25$.

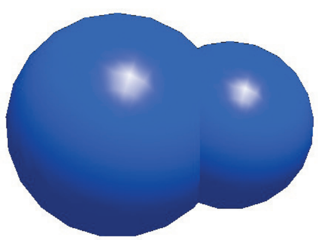


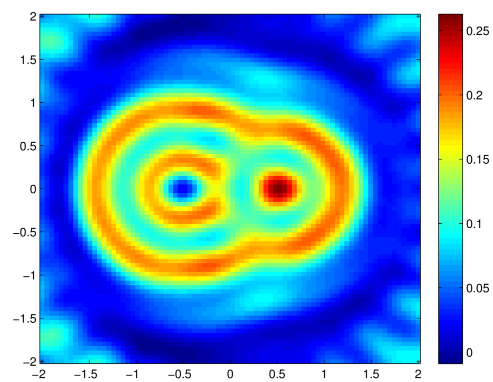
The sampling domain is $\Omega = (-3, 3) \times (-3, 3)$ with the sampling grid 201×201. The probe wave wavenumber $k = 4\pi, N_s = N_r = 128$, and $R_s = R_r = 10$.

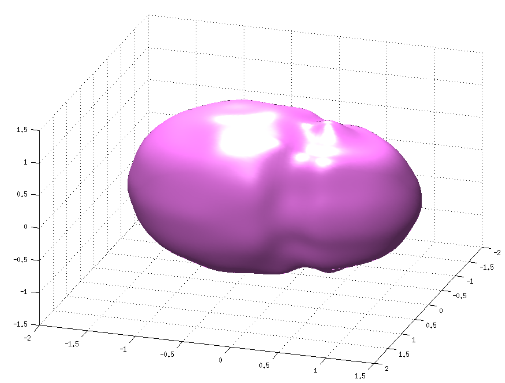
Literature Remarks

1. **Full space imaging: acoustic [Chen-C.-Huang, 2013], electromagnetic [Chen-C.-Huang, 2013], elastic [C.-Huang, 2015]; closed waveguides: [C.-Huang, 2015]; half-space: [C.-Huang, 2015];**

2. **Phaseless: acoustic [C.-Huang, 2016], electromagnetic [C.-Huang, 2016], half-space acosutic colorblue[C.-Fang-Huang, 2017]**







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Predictive modeling requires the error control of numerical methods!

THANK YOU!