Triangle and Beyond

Xingwang Xu

Nanjing University

October 10, 2018

1. A formula for compact surfaces

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2. Chern-Gaussian-Bonnet formula for Riemannian manifolds

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3. Generalization to complete Riemann surfaces

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6. Final Remarks

triangle

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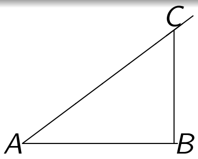

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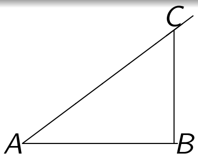

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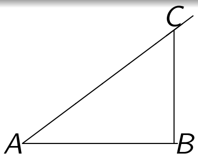

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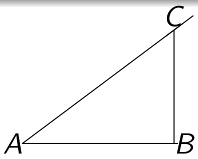

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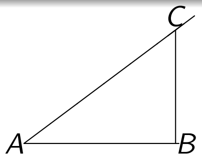

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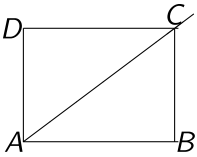
- $2\pi\chi(\Delta) =$ the sum of the exterior angles

polygon

For a polygon:



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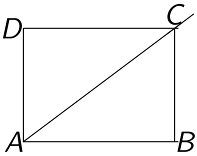

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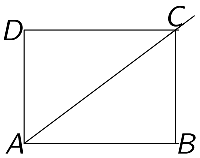

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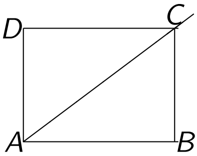

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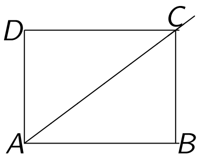

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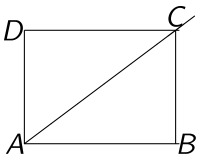

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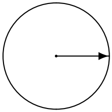
- Add more edges, i.e., to do triangulation (easy to see: χ does not change.)
- $2\pi\chi$(polygon) $=$ the sum of the exterior angles

circle

For a disc:



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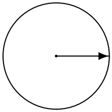

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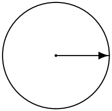

- The total sum of the infinitesimal exterior angles equals 2π
- No definition of the interior angles

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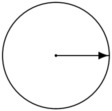

- The total sum of the infinitesimal exterior angles equals 2π
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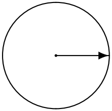

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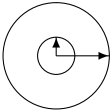
- χ(disc) $=$ the number of vertexes - the number of edges $+$ the number of faces $= 1$ (by triangulation)
- $2\pi\chi$(Disc) $=$ the sum of the infinitesimal exterior angles

domain with a hole

For annulus domain:



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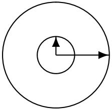

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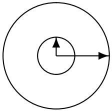

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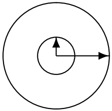

- Total sum of the exterior angle equals 0
- $\chi(\text{Annulus}) = 0$
- Still have $2\pi\chi =$ the total sum of the exterior angle

general domain

Theorem: Let Ω be a planar domain with smooth boundary $\partial\Omega$. Let k be the geodesic curvature and ds be the arc length element. Then we have

$$\chi(\Omega) = \frac{1}{2\pi}\int_{\partial\Omega} k ds.$$

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Remark: $\chi(\Omega)$ is counted according to the triangulation as before. In the case of triangle, we notice that the boundary curve is not smooth and $k = 0$ on edge, that is why we count the exterior angles.

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This was done by Gauss in 1827. Gauss shows that this is intrinsic, determined solely its line element(ds^2).

Euler number

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It was Euler who realized that this algebraic sum is very important number of the surface, although before him, there were several people noticed this number. This is called Euler number.

Gauss-Bonnet formula

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Theorem: If M is a compact surface without boundary and g is the metric on M and K is Gaussian curvature of g, then $\chi(M) = \frac{1}{2\pi}\int_M K d\sigma$ where $d\sigma$ is area element of g.

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Clearly here K is assumed to be constant on whole triangle. Bonnet (1848) generalized this result to the case that the triangle may not be geodesic triangle. In this case, the geodesic curvature on boundary should appear in such formula.

general formula

More general formula: If Ω is a smooth domain with piecewise smooth boundary curve $\partial\Omega$. Relative to the metric on Ω, we can define the geodesic curvature of $\partial\Omega$, denote it by k. Let α_i be the exterior angle at the non-smooth point p_i.

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$$2\pi\chi(\Omega) = \sum_i \alpha_i + \int_{\partial\omega} kds + \int_\Omega Kda.$$

This general formula covers the triangle and polygon cases. The importance here is to connect the local information with global information.

local theory

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- Riemann's lecture "On the hypotheses that lie at the foundations of geometry", delivered on 10 June 1854, became a classic of mathematics, now called Riemannian Geometry.

local theory

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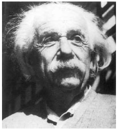
- The second part of Riemann's lecture posed deep questions about the relationship of geometry to the world we live in. He asked what the dimension of real space was and what geometry described real space.

local theory

Albert Einstein
German Scientist(1879-1955)



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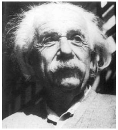

Riemann's idea had not been completely understood for at least SIXTY years until Einstein toke it up and use it in his general relativity. Before that they was one piece work done by Elwin Bruno Christoffel (1829-1900) and Gregorio Ricci-Curbastro (1853-1925), now called Levi-Civita(Ricci's student, full name Tullio Levi-Civita, 1873-1941) connection which is first order derivative of metric tensor while Riemann's curvature tensor is second order derivative of the metric tensor.

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- For any closed manifold the sum of the indices of a generic vector field is a topological invariant, namely, the Euler characteristic. This is first proved by Solonmon Lefschetz (Russian mathematician, 1884-1972).
- Generalized Gauss-Bonnet formula to compact hyper-surfaces in Euclidean space.

global theory

Andre Weil
French Mathematician(1906-1998)

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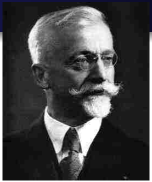
Weil and his coauthor C. B. Allendoerfer(1911-1974, American Mathematician) jointly proved Gauss-Bonnet formula for Riemann polyhedra in 1943. Their method depends on the extrinsic geometry. Namely, they embedded such manifold into higher co-dimensional Euclidean space and use the geometry of Euclidean space to get the proof. Still unsatisfied since this is a intrinsic formula. It should have a pure intrinsic proof!

global theory



Elié Cartan
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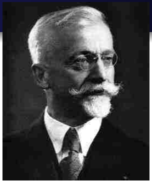

- C. B. Allendoerfer had proved Gauss-Bonnet theorem for Closed oriented Riemann manifolds in Euclidean space (1940) (extrinsic proof)

global theory

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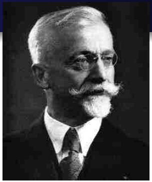

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- W. Fenchel(German Mathematician, 1905-1988) had another proof(1940) (also extrinsic proof)
- E. Cartan developed exterior differential forms and moving frame method to study differential geometry.

global theory

Shiing-shen Chern
Chinese Mathematician(1911-2004)

global theory

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Chern was visited E. Cartan and learnt his moving frame method. At the time, he might be the only mathematician who can understand Cartan's method. Chern used this method to give a beautiful intrinsic proof for Gauss-Bonnet theorem in 1944. A year later, he generalized the formula to compact manifolds with boundary. This is what now we call Chern-Gauss-Bonnet formula.

Chern's work

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Those characteristic classes are very important for the study of the vector bundle and complex manifolds.

complete manifolds

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3. Every geodesic closed ball $\bar{B}(m, r)$ is compact.
4. Every bounded closed set is a compact set.

Hopf-Rinow theorem

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And a consequence of this theorem says that every two points p, q on such manifold can be connected by a geodesic which realized the distance between p and q.

Cohn-Vossen's theorem

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Theorem (1935) If the Gaussian curvature of a complete open Riemann surface M with analytic metric is absolutely integrable, then the following inequality holds:

$$\int_M K dv_M \leq 2\pi\chi(M) \tag{1}$$

where $\chi(M)$ is the Euler number of M and K is the Gaussian curvature of the metric.

Huber's generalization

A. Huber (German mathematician, ETH)(1957): (suggested by H. Hopf) extended this inequality to metrics with much weaker regularity. More importantly, he proved that such surface M is conformally equivalent to a closed surface with finitely many punctures.

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His method is to study subharmonic functions on $\mathbf{R}^2$ and their difference. Clearly it is analytic oriented. And also the two dimensional complete Riemann surfaces have simple structure, namely compact manifolds with finitely many points removed if the total integral of the curvature is finite.

Finn's identity for complete surface

R. Finn (Stanford, 1965): For a fairly large class of complete surfaces, the deficit in Cohn-Vossen's inequality is related to isoperimetric ratio in the following formula:

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where the sum is taken over each end of M and ν_j on each end is defined by

$$\nu = \lim_{r\to\infty} \frac{L^2(r)}{4\pi A(r)}$$

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R. Finn (Stanford, 1965): For a fairly large class of complete surfaces, the deficit in Cohn-Vossen's inequality is related to isoperimetric ratio in the following formula:

$$\chi(M) - \frac{1}{2\pi}\int_M K dv_M = \sum_j \nu_j$$

where the sum is taken over each end of M and ν_j on each end is defined by

$$\nu = \lim_{r\to\infty} \frac{L^2(r)}{4\pi A(r)}$$

where $L(r)$ is the length of the boundary circle $\partial B_r = \{|x| = r\}$ and $A(r)$ is the area of the annular region $B(r)\backslash K$ for any compact set K.

Poor's extension for complete four manifold

W. A. Poor (1974) (only paper written by this author according to mathscinet): Study the case for Cohn-Vossen's inequality in dimension 4 for non-negative curved complete manifolds

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- A complete Riemannian manifold M of nonnegative curvature is diffeomorphic to the normal bundle of its soul (a concept due to M. Gromov and J. Cheeger)
- For oriented M of dimension 4, the total curvature is bounded between zero and the Euler characteristic of M. Indeed, assuming the truth of the algebraic Hopf conjecture, it is shown that this is true for oriented M of any even dimension.

Walter's work

R. Walter (German mathematician)(1975): also generalize this inequality to dimension 4.

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Let γ be constant times the Pfaffian form of the curvature form and Ω be a locally convex compact subset of an oriented Riemmannian manifold of dimension $2n$, then

$$\int_{\Omega} \gamma \leq \chi(\Omega),$$

if $n \leq 3$ and the sectional curvature is non-negative along the boundary $\partial\Omega$, otherwise it is still true if the curvature operator is positive semi-definite.

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2. If $n \geq 3$ above inequality also holds if we assume the curvature operator is positive semi-definite.

Remark: The first result here is the generalization of the well-known result of Chern and Milnor to the noncompact case. And both consequences have been shown in previous mentioned work by W. A. Poor.

Greene-Wu's work

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This paper is mainly concern with the existence of C^∞ strictly convex functions on a Riemannian manifold. Above quotation is just one of the consequences of the general results they proved. Other, for example includes the conclusion on Stein manifolds under certain assumptions.

Cheeger-Gromov's work

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- Conclusion: the integrals of characteristic forms (the Gauss-Bonnet form) over a manifold M are always convergent
- The limit values are referred to as geometric characteristic numbers and denoted by $\chi(M, g)$
- Several examples: without those extra assumptions, the conclusion is not true in general

Yau's question

Recent Development on this problem: S. T. Yau in his famous problem section (problem 11.) asked how to generalize Cohn-Vossen's inequality or Finn/ Huber's identity to higher dimension.

Chang-Qing-Yang's work on $\mathbf{R}^4$

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$$\mu = \lim_{r\to\infty} \frac{[vol(\partial B_r(0))]^{4/3}}{4(2\pi^2)^{1/3} vol(B_r(0))}.$$

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That is, we have:

$$1 - \frac{1}{8\pi^2}\int_{\mathbf{R}^4} Q_4 e^{4w} dx = \mu.$$

Chang-Qing-Yang's work on M^4

Theorem (Chang-Qing-Yang, 2000)
Suppose M is a locally conformally flat complete $4-$manifold with only finitely many conformally flat simple ends. And suppose that the scalar curvature is non-negative at each eand, and the Q_4 curvature is absolutely integrable. Then, we have

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where in the inverted coordinates centered at each end,

$$\mu_k = \lim_{r\to\infty} \frac{V_3(\partial B_r(0))^{4/3}}{4(2\pi^2)^{1/3} Vol(B_r(0)\backslash B_1(0))}.$$

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where E is the trace-less Ricci tensor.
While the integrand of Gauss-Bonnet-Chern is equal to $|W|^2 + Q_4$ where W is just Weyl tensor. In conformal flat case, $W \equiv 0$.

remark

■ Definition of Q_n curvature and their associated differential or pseudo-differential operators.

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P_3 is just boundary operator of Δ^2

$n = 4$: Q_4 curvature defined by Paneitz with P_4 as I pointed out before. I should remind you that the definition of Q_4 has different version. But they are the same except a constant multiple. We take this version just for convenience of the statement.

remark

■ $n \geq 5$: It is almost impossible to explicitly write them out both for Q_n and for P_n. For n even, Q_n is defined by T. Branson while P_n was found by C. Robin Graham, R. Jennes, L.J. Mason, and G.A.J. Sparling around 10 year ago. However, very recently, C. Robin Graham and M. Zworski studied the Poincaré metrics on $M \times [0,1]$ and their scattering matrix recapture the Q_n as well as P_n when n is even. C. Fefferman and C. Robin Graham further studied the case that n is odd and found both Q_n and P_n in just one place.

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■ Promising theory and deserves further study. Also relate to Witten theory regarding Anti-de Sitter space/CFT correspondence which is out of my control.
Roughly speaking, consider the manifolds as the boundary of one dimensional higher manifolds N. If N has a very nice metric structure, say Einstein metric with negative cosmological constant. There exists a well defined defining function for the boundary. Then the total volume should be infinity. However the volume of N subtracting the $\epsilon-$neighborhood should be finite and it depends on ϵ. We can take Taylor expansion in ϵ, certain coefficient in this expansion will depend only the conformal structure of the boundary. This coefficient (so called the normalized volume) is related to the quality we discussed above.

normal metric on $\mathbf{R}^n$

First consider the complete metric $e^{2w}g_0$ on $\mathbf{R}^n$ with absolutely integrable Q_n curvature and nonnegative scalar curvature at infinity. We shown that Finn's formula holds with ν replacing by iso-perimetric ratio μ defined as follows:

$$\mu = \lim_{r\to\infty} \frac{[vol(\partial B_r(0))]^{n/(n-1)}}{n(\omega_n)^{1/(n-1)} vol(B_r(0))}.$$

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That is, we have:

$$1 - \frac{2}{(n-1)!\omega_n} \int_{\mathbf{R}^n} Q_n e^{nw} dx = \mu.$$

Q_n- curvature

Notice that in conformally flat case, the Q_n-curvature is simply defined by

$$Q_n(x) = e^{-nw(x)}[(-\Delta)^{n/2} w](x).$$

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Remark: We do not need to assume the dimension n is even.

mixed volume

It is well-known that in higher dimension, except above iso-perimetric ratio, there are also the concept of mixed volumes for a convex set in $\mathbf{R}^n$, in particular, we can define them for the boundary of the ball $B_r(x_0)$ in $\mathbf{R}^n$.

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Such mixed volume concept has been considered by N. Trudinger around 15 years ago.
For the conformal metric $g = e^{2w} g_0$, it is defined as

$$V_n(r) := \int_{B_r(0)} e^{nw} dx;$$

iso-perimetric ratio

and

$$V_k(r) := \frac{1}{n}\int_{\partial B_r(0)} r^{k-n+1}(1 + r\frac{\partial w}{\partial r})^{n-k-1} e^{kw}\, d\sigma;$$

for $k = 1, 2, \cdots, n-1$.

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And now for all $1 \le j \le n-1$ and $1 \le k \le n-j$, we define the iso-perimetric ratios to be:

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And now for all $1 \leq j \leq n-1$ and $1 \leq k \leq n-j$, we define the iso-perimetric ratios to be:

$$C_{k,k+j}(r) := \frac{V_k^{(k+j)/[j(n-1)]}}{(n\omega_n)^{1/(n-1)} V_{k+j}^{k/[j(n-1)]}}.$$

statement I

Theorem: Suppose that $e^{2w}g_0$ is a smooth complete metric on $\mathbf{R}^n$ with absolutely integrable Q_n-curvature and non-negative scalar curvature at infinity. Then

$$\int_{\mathbf{R}^n} Q_n e^{nw} dy \leq \frac{(n-1)!\omega_n}{2},$$

and

$$\begin{aligned} \alpha &= \lim_{r\to\infty} C_{n-1,n}(r) \\ &= 1 - \frac{2}{(n-1)!\omega_n}\int_{\mathbf{R}^n} Q_n e^{nw} dy \geq 0. \end{aligned}$$

statement I, continued

Theorem: Furthermore, if $\alpha > 0$, then we also have

$$\lim_{r\to\infty} C_{k,k+j}(r) = \alpha,$$

for all k and j satisfy above restrictions.

local version

Due to the limit understanding on Gauss-Bonnet-Chern formula for higher dimensional manifolds, we have no general result for all dimensions in this case. And as we all know that there is no such formula for odd dimensions, we can only have a result in conformal flat case.

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General local conformally flat manifolds might still be too complicated, specially their ends structure. Here our discussion will be focused on some subclass of those manifolds.

locally conformal flat complete manifolds

Definition: (Simply connected, conformally flat Manifolds with conformally flat simple ends)

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$$M = N \cup \{\cup_{k=1}^{l} E_k\}$$

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Definition: (Simply connected, conformally flat Manifolds with conformally flat simple ends)
Suppose (M, g) is given as

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where (N, g) is a compact locally conformal flat manifold with boundary

$$\partial N = \cup_{k=1}^{l} \partial E_k$$

definition, continued

and each E_k is a conformal flat simple end of M; that is,

$$(E_k, g) = (\mathbf{R}^n \backslash B_1(0), e^{2w_k} g_0),$$

for some function w_k, where $B_1(0)$ is the unit ball in $\mathbf{R}^n$.

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for some function w_k, where $B_1(0)$ is the unit ball in $\mathbf{R}^n$.
Then we say that (M, g) is a complete conformal flat $2m-$manifold with a finite number of conformal flat simple ends.

statement II

With our discussion for complete conformal metric on $\mathbf{R}^n$, and its local version on $\mathbf{R}^n \backslash B_1(0)$, we can show the following Theorem.

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Basic assumptions: Suppose that (M, g) is a complete conformal flat $n = 2m-$manifold with a finite number of conformal flat simple ends. Suppose that

(a) The scalar curvature is non-negative at infinity on each end;

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With our discussion for complete conformal metric on $\mathbf{R}^n$, and its local version on $\mathbf{R}^n \backslash B_1(0)$, we can show the following Theorem.

Basic assumptions: Suppose that (M, g) is a complete conformal flat $n = 2m-$manifold with a finite number of conformal flat simple ends. Suppose that

(a) The scalar curvature is non-negative at infinity on each end;

(b) Its Q_n curvature is absolutely integrable; that is,

$$\int_M |Q_n| dv < \infty.$$

statement II, continued

Conclusion:

$$\chi(M) - \frac{2}{(n-1)!\omega_n} \int_M Q_n dv = \sum_{k=1}^{l} \mu_k,$$

where

$$\mu_k = \lim_{r \to \infty} \frac{[\int_{\partial B_r(0)} e^{(n-1)w_k} d\sigma]^{n/(n-1)}}{n(\omega_n)^{1/(n-1)} \int_{B_r(0)} e^{nw_k} dx}.$$

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In particular, we have, for such manifolds,

$$\chi(M) \geq \frac{2}{(n-1)!\omega_n}\int_M Q_n dv.$$

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- This result provides some inside information for what Chern-Gauss-Bonnet's formula really means.
- $\mathbf{R}^n$ version formula has been reproved recently by C. B. Ndiaye and J. Xiao, preprint 2008.

negative sectional curvature case

- We notice that J. Cao and F. Xavier show that the Euler number $\chi(M^{2n})$ of a compact Riemannian manifold M^{2n} of non-positive curvature which is homeomorphic to a KÅ¡ahler manifold must satisfy the inequality

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- Does this have something to do with our case? Notice that, in this case, the universal covering space is diffeomorphic to $\mathbf{R}^n$. So it will have only one end. In our study it requires conformal flat end.

positive sectional curvature case

- This case should be easier since, according to Cheeger and Gromoll's famous splitting theorem, M can be written as $\bar{M} \times \mathbf{R}^k$ which has simple end structure.

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- Notice that Greene and Wu's result in dimension 4 is within this category.

hyper-surfaces

Dillen, Franki and Kemhnel, Wolfgang(Total curvature of complete submanifolds of Euclidean space Tohoku Math. J. (2) 57, no. 2 (2005), 171-200)

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- For higher dimensional hyper-surfaces, the curvature defect can be positive, zero, or negative, depending on the shape of the ends "at infinity".
- An explicit example of a 4-dimensional hyper-surface in Euclidean 5-space where the curvature defect is negative is given. The direct analogue of the Cohn-Vossen inequality does not hold.

hyper-surfaces, continued

- For open hyper-surfaces with cone-like ends, the total curvature is stationary for deformation $\Leftrightarrow$ each end has vanishing Gauss-Kronecker curvature in the sphere "at infinity". For this case of stationary total curvature we prove a result on the quantization of the total curvature.

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- Does our result have anything to do with this?

the end

Thank You Very Much!